Conventions and Social Mobility in Bargaining Situations

by

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Abstract

This paper studies the evolution of a population whose members use their social class to coordinate their actions in a simple tacit bargaining game. In the spirit of Rosenthal and Landau [1979], we interpret the equilibrium behaviours that the players may adopt, as a function of their class, as *customs*. Players may change their class depending on the outcome of the game, and may also change their custom, as a result of some learning process. We are interested in the characterization of the fixed points of the adjustment process over the space of classes and customs from a distributional point of view. We find that, although any custom (when it operates alone) generates the same limiting class distribution as any other, these limiting distrbutions can be ranked with respect of their *mobility*. If players are allowed to change their custom when they find it unsatisfactory, then social mobility appears to be the key variable to predict the type of custom which will predominate in the long run even though, in general, no one custom is dominant. In particular, customs which promote social mobility appear to exhibit, in all the cases we have analysed, stronger stability properties.

1. Introduction

There are many economic situations in which informal means are employed to execute mutually beneficial agreements. In such cases, some social variable (call it *class*, or *reputation)* may help the agents to coordinate their actions on an equilibrium of the game they are playing. The notion of *convention*, often used to describe these equilibria, may then involve some sociological background: a particular behaviour may have no intrinsic merit, but is selected on the basis of some social or cultural link among the players[1]. The role of these social variables may be even more important in those situations where, for such an equilibrium to be implemented, different agents are required to adopt different behaviours (and receive, in return, different rewards). In this case, the social context may in fact determine who is supposed to do what (and, consequently, who deserves the lion's share).

The society we have in mind is modelled by a constant utility flow which is to be allocated, in each time period, by means of a simple bargaining scheme between two players, randomly selected from the population. Each player has to choose, simultaneously, whether to *defect* (requiring the biggest share for herself) or to *cooperate* (accepting the division proposed by the opponent). If both players cooperate, then the pie is equally divided; if both defect, then the size of the pie will be substantially reduced, as a result of the negotiation breakdown. The only information available to each player is the opponent's *class*, that is, a signal from which it can be partially deduced the opponent's past behaviour in the stage game. We shall assume that the strategic choice of the two players is conditioned only on this information. The outcome of the stage game may modify the class of the players, who are then placed back in the original population. In the following time period, other two players will be paired at random, and so on.

Rosenthal and Landau [1979] (R&L hereafter) explore, under similar conditions[2], how some behavioural patterns, which they call *customs*, may influence the long-run distribution of plays

[1] In his seminal contribution on the study of conventions, Schelling [1963]argues that: "...The force of many rules of etiquette and social restraint ... seems to depend on their having become "solutions" to a coordination game..." (p. 91).

[2] See Rosenthal [1979] for a formal description of the general framework.

of the population game. In their paper, these customs are described as ”*...possible decision rules which members of the society might unanimously employ to determine their moves in the game...”.*[1] Two properties characterize a custom under their perspective:

a) it uniquely determines the players' behaviour in the stage game;

b) such behaviour must be self-enforcing, in the sense that it must be justified, from the players' viewpoint, on the ground of some rationality assumption.

This definition clearly recalls what economists are now accustomed to call *conventions,* with reference to the flourishing stream of research in the recent game-theoretic literature which studies coordination games.[2] Behind this analogy stands the fact that each player faces a symmetric situation characterized by multiple equilibria. However, unlike a pure coordination setting, in the stage game we have just described, the players rank the various equilibrium outcomes differently: the selection of a particular custom can then be observed from a *distributional* point of view, since a better bargain for a player implies less for the opponent.

In R&L's model, the social variable upon which players condition their choice is termed *reputation* (higher reputation signifying tendency to defect). Moreover, they assume that each individual in the population follows the same custom: different customs generate different limiting distributions, which are then compared in terms of their efficiency properties. Intuition suggests, the authors claim, that customs which prescribe cooperating against a player with higher reputation (seemingly more prevalent in real-life bargaining situations) might also be justified on efficiency grounds, once they minimised the social loss generated in equilibrium. However, commenting on their results, R&L admit that, in their model ”...this has proved not to be the case...” (p. 234), since the social ranking of the equilibrium customs depends crucially on how reputation is formally defined.

Our model differs from R&L's original formulation in (at least) two respects. First, we assume that the social variable (we call it *class*) is directly linked to the payoffs received during the past

[1] See Rosenthal and Landau [1979], p. 234.

[2] See, among others, Kandori *et al.* [1993] and Young [1993]

history of the game. In particular, we shall assume that the class of a player is simply the payoff she received the last time she has been called to play. We justify this assumption by interpreting the class as a signal of each individual's wealth.

Moreover (and more crucially), we do not necessarily assume that a unique custom is commonly shared in the society. Instead, we allow the possibility that the agents hold different customs. This feature of our model opens the possibility of modelling a *learning* process: players may in fact change their custom if they find it somehow unsatisfactory.

We design the learning process at two different, and somehow complementary, levels. We consider first what we call *coordination learning:* players holding different customs may fail to coordinate their actions. We therefore model a procedure which leads the players to revise their custom as a result of a disequilibrium play. In addition, we introduce a further type of learning, which we call *aspiration learning.* After the stage game has been played, each player compares her own payoff to some threshold value by which we take to be an estimate of a "satisfactory" outcome of the strategic interaction. Whenever this aspiration level is not reached, a player is assumed to modify her custom with positive probability.

Our coordination and aspiration learning schemes allow some individual feed-back to the social outcome induced by each custom; one of the aims of the paper is to explore how this feed-back interacts with the social pressures generated by our custom society. [1]

The remainder of the paper is arranged as follows. Section 2 describes the main features of the model. Section 3 develops the formal theory on which our analysis is based. Following

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[1] There are many references, in the macroeconomic litterature on income distribution, which stress the role of social variables in the determination of the income distribution in society. Becker and Tomes [1979], for example, point out that: "...The concept of endowment is also a fundamental part of our analysis. Children are assumed to receive endowments of capital that are determined by the reputations and "connections" of their families, the contribution to the abilities, race, and other characteristics of children from the genetic constitutions of their families, and the learning, skills, goals, and other "family commodities" acquired through belonging to a particular family culture..." (p. 1158).

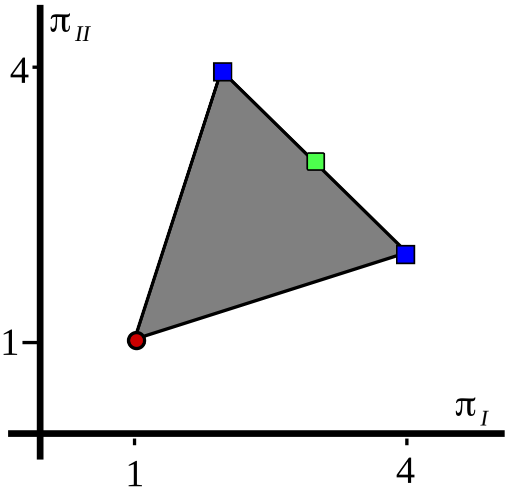
R&L, section 4 assumes that only one custom is adopted by the entire population, and explores the asymptotic properties of the limiting class distribution, under different customs. In this respect, we find (consistently with R&L) that the limiting class distribution under any particular custom *is exactly the same*. We interpret this result as follows. If a custom allows the players to coordinate on one of the Nash equilibria of the game, and the class of a player is the payoff received, the limiting class distribution will concentrate most of its mass on the classes which correspond to the payoffs that the players get when a pure strategy Nash equilibrium is played, no matter how this coordination takes place (i.e. regardless of the custom which is actually established).

It is important to notice that it does not follow from the above result that, once the equilibrium distribution has been reached, the same players will stay in the same class forever after. On the contrary, each custom is characterised in equilibrium by a complex, but balanced, network of flows among classes. Section 5 explores the properties of a society in which only on custom is available from this perspective, interpreting these flows as measures of *social mobility*.

We then move to a situation where different customs are present at the same time within the population. Section 6 explore the simplest possible case (that is, a two-custom society); section 7 considers the case of a society in which all possible customs may be present. If players are allowed to change their custom through learning processes in the way we described, then social mobility appears to be the key variable for predicting the type of custom which will predominate in the long run. In particular, even thuogh no custom is dominant, customs which promote social mobility appear to exhibit stronger stability properties in all the cases we have analysed. A final section devoted to additional remarks concludes, followed by four sections of appendix containing the most elaborate proofs.

2. The basic model

We deal with a market economy characterized by a constant utility flow which is to be allocated in each time period within a large, but finite population of N players. At each point in time two individuals are drawn at random and sequentially from the population to play the symmetric normalform game of Figure 2.1, known in the literature as *chicken*, which tries to capture the intuition of a simple tacit bargaining situation. The game is characterized by two asymmetric Nash Equilibria in pure strategies, namely (C, D) and (D, C), and a symmetric Nash Equilibrium in which each pure strategy is played with equal probability.[4]



	C	D
C	3,3	2,4
D	4,2	1,1

FIGURE 2.1. *The Class Game.*

At any giventime, the *type* of each player is characterized by a *class* and a *custom* as explained in the following two definitions.

Definition 2.1. The class of a player is simply the payoff received in the last round she has been called to play.

We interpret the class as a measure of the stock of wealth inherited from the past history of the game. Let $\mathcal{C} = \{1, 2, 3, 4\}$ be the set of classes and $S = \{C, D\}$ the strategy set in the Class Game. The custom of a player determines her behaviour in the Class Game:

Definition 2.2. A generic *custom* k is a function $k : \mathcal{C} \times \mathcal{C} \to \Delta(S)$, which satisfies the following conditions:

$$\left.\begin{array}{l} k(c, c') = 0 \text{ or } 1 \text{ when } c = c' \\ k(c, c') = k(c', c) = 1/2 \text{ when } c = c' \\ k(c, c') = 1 - k(c', c) \end{array}\right\} \qquad (2.1)$$

[4] Following a well-established tradition, we label the two strategies available C (for *cooperate*) and D (for *defect*), the latter identifying the *minmax* strategy of the Class Game.

In words: if two players follow the same custom, they are able to coordinate their actions on one of the Nash equilibria of the Class Game.[7] In particular, if they belong to different classes, the custom tells them who is supposed to cooperate and who is supposed to defect. In the case of a play between two players of the same class, given the fact that they are absolutely indistinguishable for each other, the custom still assures that an optimal behaviour, even if only *ex ante,* is selected; namely the symmetric mixed-strategy Nash equilibrium. The interpretation is the following: the players aim to maximize their class (and therefore their share of the utility pie), and use their current class as a signal for their opponents, who condition (via the custom they follow) their behaviour on that signal. Each player can observe the class of her opponent (but not his custom), and reacts according to the dictate of her own custom, which acts as a signal extracting device.[8]

Definition 2.2 allows for the possibility of 64 different customs, since there are 6 possible encounters between players of a different class, and two choices for each player (and therefore there are $2^6 = 64$ different customs). From Definition 2.2, it is clear that each of the 64 possible customs is completely specified by the list of six numbers, either zero or one,

$$\kappa = \{k(1,2), k(1,3), k(1,4), k(2,3), k(2,4), k(3,4)\}.$$

indicating the pure strategy selected by the row player in the event of being matched with an opponent belonging to a different class. Taking as alphabet the pair $\{0,1\}$, we may therefore number the customs in their lexicographic ordering, as shown in Table 2.1.

[7] Given each player can choose only between two pure strategies in the Class Game, Definition 2.2 interprets the mixed strategy $k(c, c')$ as the probability of defecting. The probability of cooperating is then uniquely determined as $1 - k(c, c')$.

[8] We confine our attention to the range of possible behaviours represented by the set of customs, as described in Definition 2.2. This restriction is not innocent: we do not consider here a wide range of alternative behaviours which may affect the dynamics of the system. We justify this focus by arguing that a behaviour which is *internally coherent* (represented by a custom) may not be *consistent*, given the fact that the custom followed by each agent is not publicly known, and the customs used by different players may be lead to a non-Nash outcome. In this way we introduce non-equilibrium behaviour in the model, while keeping its complexity under control.

#	κ	#	κ	#	κ	#	κ
1	$\{0,0,0,0,0,0\}$	17	$\{0,1,0,0,0,0\}$	33	$\{1,0,0,0,0,0\}$	49	$\{1,1,0,0,0,0\}$
2	$\{0,0,0,0,0,1\}$	18	$\{0,1,0,0,0,1\}$	34	$\{1,0,0,0,0,1\}$	50	$\{1,1,0,0,0,1\}$
3	$\{0,0,0,0,1,0\}$	19	$\{0,1,0,0,1,0\}$	35	$\{1,0,0,0,1,0\}$	51	$\{1,1,0,0,1,0\}$
4	$\{0,0,0,0,1,1\}$	20	$\{0,1,0,0,1,1\}$	36	$\{1,0,0,0,1,1\}$	52	$\{1,1,0,0,1,1\}$
5	$\{0,0,0,1,0,0\}$	21	$\{0,1,0,1,0,0\}$	37	$\{1,0,0,1,0,0\}$	53	$\{1,1,0,1,0,0\}$
6	$\{0,0,0,1,0,1\}$	22	$\{0,1,0,1,0,1\}$	38	$\{1,0,0,1,0,1\}$	54	$\{1,1,0,1,0,1\}$
7	$\{0,0,0,1,1,0\}$	23	$\{0,1,0,1,1,0\}$	39	$\{1,0,0,1,1,0\}$	55	$\{1,1,0,1,1,0\}$
8	$\{0,0,0,1,1,1\}$	24	$\{0,1,0,1,1,1\}$	40	$\{1,0,0,1,1,1\}$	56	$\{1,1,0,1,1,1\}$
9	$\{0,0,1,0,0,0\}$	25	$\{0,1,1,0,0,0\}$	41	$\{1,0,1,0,0,0\}$	57	$\{1,1,1,0,0,0\}$
10	$\{0,0,1,0,0,1\}$	26	$\{0,1,1,0,0,1\}$	42	$\{1,0,1,0,0,1\}$	58	$\{1,1,1,0,0,1\}$
11	$\{0,0,1,0,1,0\}$	27	$\{0,1,1,0,1,0\}$	43	$\{1,0,1,0,1,0\}$	59	$\{1,1,1,0,1,0\}$
12	$\{0,0,1,0,1,1\}$	28	$\{0,1,1,0,1,1\}$	44	$\{1,0,1,0,1,1\}$	60	$\{1,1,1,0,1,1\}$
13	$\{0,0,1,1,0,0\}$	29	$\{0,1,1,1,0,0\}$	45	$\{1,0,1,1,0,0\}$	61	$\{1,1,1,1,0,0\}$
14	$\{0,0,1,1,0,1\}$	30	$\{0,1,1,1,0,1\}$	46	$\{1,0,1,1,0,1\}$	62	$\{1,1,1,1,0,1\}$
15	$\{0,0,1,1,1,0\}$	31	$\{0,1,1,1,1,0\}$	47	$\{1,0,1,1,1,0\}$	63	$\{1,1,1,1,1,0\}$
16	$\{0,0,1,1,1,1\}$	32	$\{0,1,1,1,1,1\}$	48	$\{1,0,1,1,1,1\}$	64	$\{1,1,1,1,1,1\}$

TABLE 2.1. *Numbering customs.*

Table 2.1 lists every possible behaviour allowed by Definition 2.2: from κ_1, which always prescribes the lower class player to cooperate against a higher class opponent, to the opposite extreme κ_{64}, in which the lower class player always defect, together with every possible combination between the two.

We assume that all the players follow a custom (not necessarily the same) that completely characterizes their strategic behaviour in the Class Game, which in turn determines their current payoff and, therefore, their new class when they are then placed back in the original population. At the beginning of the following round, two new players will be paired at random, and so on. Loosely speaking, the above mechanism generates a dynamic over the set of classes $\mathcal{C}$; *i.e.* for each agent a in the population, there is generated a *class history*, in the form of a sequence $c_0(a), c_1(a), \ldots, c_n(a), \ldots$, with $c_t(a) \in \mathcal{C}$, and $c_n(a)$ agent a's class in round n. In the remainder of the paper, we will refer to this as the *Class Dynamic*.

Given our assumptions, at each point in time, the state of the system is identified by the vector $\mathbf{x}(t) = \{x_{(c,k)}(t)\}$ of proportions of players characterized by the class c and the custom k at time t. Denote by Ω_N the set of such states, i. e. the state space of the system. Notice that that Ω_N is a finite set: the underlying dynamic is therefore a stochastic process defined over a finite state space, the properties of which will be formally explored in the following sections.

3. Some general theory

Our analysis of the system described above will be based on the general theory developed in Seymour (1994). In this section we give a brief synopsis of those features of the theory we require. We consider a (large) population of N agents, each of whom can be any one of m possible "types", $\{1, 2, \ldots, m\}$, at any given time. [3] Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ be the vector of proportions of the population in each type. We assume that during each small time interval of length Δt, two individuals are chosen at random (without replacement) from the population. These individuals (and no others) then interact in some way (*e.g.* by playing the 2-player game described in section 2), the effect of which is to change their type. Thus, if the agents have types (i, j) before the interaction, then the interaction results in a transition $(i, j) \to (i', j')$ with some specified probability, $p(i', j'|i, j)$. After the interaction, the agents return to the population, and the process is repeated in the next time interval. The transition probabilities are assumed to satisfy

$$\sum_{i',j'} p(i', j'|i, j) = 1 \qquad \text{for each } i, j, \tag{3.1a}$$

$$p(i', j'|i, j) = p(j', i'|j, i) \qquad (\text{symmetry}). \tag{3.1b}$$

The symmetry condition simply means that the interaction outcome is unaffected by whichever of the two participants is chosen first. Condition (3.1a) also implies that the repeated process is a discrete-time Markov process on the rational lattice

$$\Omega_N \subset \Delta^{m-1} = \{\mathbf{x} \in \mathbb{R}^m \mid 0 \le x_i \le 1, \text{ and } \sum_i x_i = 1\}, \tag{3.2}$$

[3] As already mentioned, in the framework of section 2, a *type* is simply a pair (c, k), denoting the class and the custom of an individual.

consisting of those points $\mathbf{x}$ for which $N_i = x_i N$ is an integer. Here, Δ^{m-1} is the $(m-1)$-dimensional simplex in its standard embedding in $\mathbb{R}^m$. In fact, each interaction results in a state transition, $\mathbf{x} \to \mathbf{x}'$, between points in Δ_N, which, if the participants have initial types (i, j), has the form

$$\mathbf{x}' = \mathbf{x} + \nu^{ij} \Delta x, \qquad (3.3)$$

where $\Delta x = 1/N$, and $\nu^{ij} = (\nu_1^{ij}, \nu_2^{ij}, \ldots, \nu_m^{ij})$, is a vector-valued random variable, with each $\nu_a^{ij} \in \{0, \pm 1 \pm 2\}$, such that ν_a^{ij} is the change in the *number* of individuals of type a which results from an interaction between agents of types (i, j). Thus, the possible values of ν_a^{ij} are given by:

$$\nu_a^{ij} = \nu_a^{ij}(i', j') := (\delta_a^{i'} + \delta_a^{j'}) - (\delta_a^{i} + \delta_a^{j}) \qquad \text{with probability} \quad p(i', j' | i, j), \qquad (3.4)$$

where δ_a^b is the Kronecker-delta function: $\delta_a^b = 1$ if $a = b$, and $\delta_a^b = 0$ otherwise. From (3.4) we can easily compute the expected value of ν_a^{ij},

$$\bar{\nu}_a^{ij} = \sum_k \left[p(a, k | i, j) + p(k, a | i, j) \right] - (\delta_a^i + \delta_a^j). \qquad (3.5)$$

We shall be interested in the limiting, continuous-time process as $N \to \infty$ and $\Delta t \to 0$. We may think of $\Delta x = 1/N$ as the probability that a particular individual will be picked at random from the population, so that, for large N, the probability that a particular individual will participate in an interaction is $2\Delta x$ (to first order in Δx).[10] It follows that $2\frac{\Delta x}{\Delta t}$ is the (number) frequency with which a specified individual participates in an interaction. We shall take the above limits while keeping this frequency constant; *i.e.* keeping $\frac{\Delta x}{\Delta t} = c$, constant. For convenience we assume that the time scale is chosen so that $c = 1$. The result is a deterministic system on Δ^{m-1} given by the system of differential equations

$$\frac{dx_a}{dt} = \sum_{i,j} \bar{\nu}_a^{ij} x_i x_j, \qquad (3.6)$$

Equations (3.6) are derived formally in Seymour (1994), but the intuition is clear: the rate of change in x_a is the sum, for all possible type pairs, of the expected changes to type a resulting

[10] An individual has two chances of being picked, one as player-I, with probability Δx and the other as player-II, with probability $\Delta x/(1 - \Delta x)$.

from interactions between players of types (i, j), the probability with which such an interaction occurs being $x_i x_j$.

Now suppose that $P_N(\mathbf{x})$ is a probability distribution on the finite lattice Δ_N. In general, this distribution will change under the discrete-time Markov process on Δ_N. If the Markov process is ergodic, then there is a unique stationary (ergodic) distribution, $\bar{P}_N(\mathbf{x})$, such that $P_N(\mathbf{x}, n\Delta t) \to \bar{P}_N(\mathbf{x})$ as $n \to \infty$. Now, as explained in Seymour (1994), if $\lim_{N\to\infty} P_N$ is represented by a density[11], $\pi(\mathbf{x}, t)$, on Δ^{m-1}, then π satisfies the *continuity equation*

$$\frac{\partial \pi(\mathbf{x}, t)}{\partial t} + \operatorname{div}\left[f(\mathbf{x})\pi(\mathbf{x}, t)\right] = 0, \tag{3.7}$$

where $f(\mathbf{x})$ is the vector field on Δ^{m-1} given by the right hand side of (3.6). In particular, if the limiting ergodic distribution is represented by the density $\bar{\pi}$, then

$$\operatorname{div}\left[f(\mathbf{x})\bar{\pi}(\mathbf{x})\right] = 0. \tag{3.8}$$

From this we can prove

PROPOSITION 3.1[12]. Suppose the Markov process on Δ_N is ergodic for each $N \geq N_0$, and that the system (3.6) has a unique, globally asymptotically attracting equilibrium, $\bar{\mathbf{x}}$. Then the limiting ergodic distribution on Δ^{m-1}, as $N \to \infty$, is represented by the mass-point density, $\bar{\pi}(\mathbf{x}) = \delta[\mathbf{x} - \bar{\mathbf{x}}]$.

Proof. It suffices to show that the mass-point density is the unique solution of (3.8). By Proposition 5.2 of Seymour (1994), the hypotheses on $f(\mathbf{x})$ imply that any solution of (3.7) satisfies, $\pi(\mathbf{x}, t) \to \delta[\mathbf{x} - \bar{\mathbf{x}}]$ as $t \to \infty$. But, $\bar{\pi}(\mathbf{x})$ is a stationary solution of (3.7), and so $\bar{\pi}(\mathbf{x}) = \delta[\mathbf{x} - \bar{\mathbf{x}}]$. □

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[11] The limit keeps $\frac{\Delta a}{\Delta x} = 1$, so that $\Delta t \to 0$ as $N \to \infty$, yielding a continuous-time model. Also, it is not strictly necessary to assume that the limit of densities is a density, we can work with measures instead. We assume densities here only to avoid uninteresting technicalities - see Seymour (1994).

[12] A similar result, expressed in the language of sample paths, is obtained by Boylan, (1991), corollary 2.3.

4. One-custom society

The aim of this (and the next) section is to analyze the asymptotic properties of the model described in Section 2 when only one generic custom k is followed by the entire society. In this section, therefore, the dynamic will act only on a subset of states $\Omega_N^k \subset \Omega_N$ with the following properties: $x_{(c,k')}(t) = 0$ when $k' \neq k$ and $\sum_c x_{(c,k)}(t) = 1$, for all t.

Let $k \in K = \{k_1, \ldots, k_{64}\}$ be the custom used by everyone in the population. If player-I and player-II have classes i and j, respectively, then $k(i,j)$ is the probability (either 0,1 or $\frac{1}{2}$ - see (2.1)) that player-I will defect, and $k(j,i)$ is the probability that player-II will defect. As discussed in section 3, the game results in class transitions $(i,j) \to (i',j')$, and we denote by $p(i',j'|i,j)$ the probability for such a pairwise transition. These transition probabilities are easy to specify in this single custom case, and are

$$p(i',j'|i,i) = \frac{1}{4}\left(\delta^1_{i'}\delta^1_{j'} + \delta^2_{i'}\delta^4_{j'} + \delta^3_{i'}\delta^3_{j'} + \delta^4_{i'}\delta^2_{j'}\right) \qquad (4.1a)$$

$$p(i',j'|i,j) = k(j,i)\delta^2_{i'}\delta^4_{j'} + k(i,j)\delta^4_{i'}\delta^2_{j'} \qquad (i \neq j) \qquad (4.1b)$$

$$p(i',j'|i,j) = 0 \qquad \text{otherwise} \qquad (4.1c)$$

Thus, the only possible transitions are: $(i,i) \to (i',j') \in \{(1,1),(2,4),(3,3),(4,2)\}$, each with probability $\frac{1}{4}$, and, if $i \neq j$, $(i,j) \to (2,4)$ if $k(j,i) = 1$ (player-I Cooperates and player-II Defects), or $(i,j) \to (4,2)$ if $k(i,j) = 1$ (player-I Defects and player-II Cooperates).

As explained in the previous section, these probabilities determine a Markov process on the rational lattice Ω_N^k contained in the 3-dimensional simplex

$$\Delta^3 = \{\mathrm{x} = (x_1, x_2, x_3, x_4) \mid 0 \leq x_i \leq 1 \text{ and } \sum_i x_i = 1\}.$$

Here, $x_i = N_i/N$ is the proportion of the total population (of size N) in class i.

PROPOSITION 4.1. For $N \geq 4$ the one-custom Markov process defined on Ω_N^k is ergodic.

Proof. See Appendix D. □

It now follows from Proposition 3.1 and Proposition 4.1 that if the derived continuous-time, deterministic system (3.6) admits a globally stable equilibrium, $\bar{\mathbf{x}}$, then the limiting ergodic distribution is represented by the mass-point density, $\delta[\mathbf{x}-\bar{\mathbf{x}}]$. We are interested in the asymptotic properties of the class dynamic when all the individuals follow the same custom:

PROPOSITION 4.2. The system (3.6) is independent of the custom k and has a unique equilibrium, $\bar{\mathbf{x}} = \left(\frac{1}{4}(3-\sqrt{7}), \frac{1}{4}(\sqrt{7}-1), \frac{1}{4}(3-\sqrt{7}), \frac{1}{4}(\sqrt{7}-1)\right)$, which is globally asymptotically stable.

Proof. The coefficients in equations (3.6) are given by (3.5), and, using equations (4.1), we have

$$c_k^{k,k} = -\frac{3}{2} \tag{4.2a}$$

$$c_k^{i,i} = \frac{1}{2} \qquad (i \neq k) \tag{4.2b}$$

$$c_k^{i,j} = (\delta_k^2 + \delta_k^4) - (\delta_k^i + \delta_k^j) \qquad (i \neq j) \tag{4.2c}$$

Note in particular, that these coefficients are *independent of the custom*, k [this is true in equation (4.2c) because $k(i,j) + k(j,i) = 1$]. Thus, so is the deterministic dynamic (3.6), and hence, so is the equilibrium, $\bar{\mathbf{x}}$. In fact, we shall show in Appendix D that the Markov process on Ω_N^4 is independent of k.

We can now obtain an explicit form for the equations (3.6). Thus,

$$\begin{aligned}
\frac{dx_k}{dt} &= c_k^{k,k} x_k^2 + \sum_{i \neq k} c_k^{i,i} x_i^2 + \sum_{i \neq j} c_k^{i,j} x_i x_j \\
&= -\frac{3}{2} x_k^2 + \frac{1}{2}\left(||\mathbf{x}||^2 - x_k^2\right) + (\delta_k^2 + \delta_k^4)\left(1 - ||\mathbf{x}||^2\right) - \left(2x_k - 2x_k^2\right) \\
&= -2x_k + \frac{1}{2}||\mathbf{x}||^2 + (\delta_k^2 + \delta_k^4)\left(1 - ||\mathbf{x}||^2\right)
\end{aligned}$$

where, as usual, $||\mathbf{x}||^2 = \mathbf{x} \cdot \mathbf{x} = \sum_i x_i^2$. Explicitely,

$$\frac{dx_1}{dt} = -2x_1 + \frac{1}{2}||\mathbf{x}||^2 \tag{4.3a}$$

$$\frac{dx_2}{dt} = 1 - 2x_2 - \frac{1}{2}||\mathbf{x}||^2 \tag{4.3b}$$

$$\frac{dx_3}{dt} = -2x_3 + \frac{1}{2}||\mathbf{x}||^2 \tag{4.3c}$$

$$\frac{dx_4}{dt} = 1 - 2x_4 - \frac{1}{2}||\mathbf{x}||^2 \tag{4.3d}$$

We are now in a position to prove Proposition 4.2. Let $\xi = x_1 - x_3$ and $\eta = x_2 - x_4$. From equations (4.3),

$$\frac{d\xi}{dt} = -2\xi \qquad \text{and} \qquad \frac{d\eta}{dt} = -2\eta$$

Thus, $(\xi(t), \eta(t)) = (\xi_0, \eta_0)\, e^{-2t}$, so that $(\xi(t), \eta(t)) \to (0, 0)$ as $t \to \infty$. It follows that, for any equilibrium $\bar{\mathbf{x}}$, we have $\bar{x}_1 = \bar{x}_3$ and $\bar{x}_2 = \bar{x}_4$. Furthermore, the subspace, $(x_1, x_2) = (x_3, x_4)$, is invariant under the dynamic (4.3), and is globally attracting. It remains to show that there is a unique attracting equilibrium inside this subspace.

Under the above constraints we have, $||\mathbf{x}||^2 = 2(x_1^2 + x_2^2)$. Also, $\sum_i x_i = 1$, reduces to $x_1 + x_2 = \frac{1}{2}$. Thus, the dynamic inside the invariant subspace is 1-dimensional, and is determined by equation (4.3a),

$$\frac{dx_1}{dt} = -2x_1 + \left(x_1^2 + \left(\frac{1}{2} - x_1\right)^2\right) = \frac{1}{4} - 3x_1 + 2x_1^2 \tag{4.4}$$

The equilibria of (4.4) are, $\bar{x}_1(\pm) = \frac{1}{4}(3 \pm \sqrt{7})$. However, only the minus sign lies in the interval $[0, 1]$, and is therefore the only allowable solution. Clearly then, $\bar{x}_1 = \bar{x}_3 = \bar{x}_1(-)$ and $\bar{x}_2 = \bar{x}_4 = \frac{1}{2} - \bar{x}_1 = \frac{1}{4}(\sqrt{7} - 1)$. Also note that (4.4) may be written

$$\frac{dx_1}{dt} = 2\left|\bar{x}_1(-) - x_1\right| \left|\bar{x}_1(+) - x_1\right|$$

The second bracket is always positive for $x_1 \in [0, 1]$, and the first bracket is positive if $x_1 < \bar{x}_1$, and negative if $x_1 > \bar{x}_1$. This shows that $\bar{\mathbf{x}}$ is globally asymptotically attracting, and therefore completes the proof of Proposition 4.2. □

Proposition 4.2 tells us that the limiting class distributions of the 64 customs coincide, and concentrate most of their mass between classes 2 and 4, the payoffs of the pure strategy Nash equilibrium. We provide, for illustrative purposes, the following histogram:

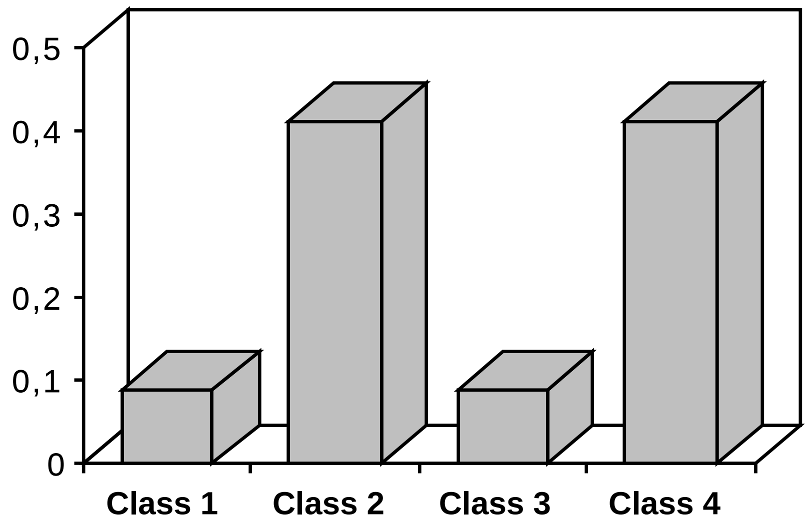

FIGURE 4.1. *The limiting class distribution under a generic custom k.*

From Proposition 4.2, we can easily calculate the expected payoff $\bar{\pi}$ of a single Class Game, given the equilibrium distribution $\bar{\mathbf{x}}$:

$$\bar{\pi} = \sum_{c \in C} c\,\bar{x}_c = \frac{1}{2}(\sqrt{7} + 3) = 2.82288 \qquad (4.5)$$

5. Social mobility

We can read the results of the previous section in the following way. In the spirit of Schotter [1981], we can interpret the game of *chicken* as an *inequality preserving* social institution, given the distributional effects associated with any self-enforcing class profile[15]. From this standpoint, it would be surprising if the limiting class distribution did not reflect the strategic features of the stage game which generates it. On the other hand, little is lost in equilibrium (since the proportion of plays in which the utility pie is not allocated in full is relatively small); from this perspective, the limiting distribution $\bar{\mathbf{x}}$ can hence be considered as a measure of the efficiency generated by the adoption of a custom (whatever it is).

However, it is important to note that, even if the limiting distribution is the same under each

[15] Here we restrict our attention to pure strategy Nash equilibria, for which the coordinating role of a custom is fully effective.

custom, it does not follow that, at each point in time, the *same* individuals belong to the same class. On the contrary, the equilibrium flows between classes may well differ in magnitude from custom to custom, with only the overall proportion remaining, on average, constant. In fact, each custom is characterised, in equilibrium, by a complex, but balanced, network of flows among classes. In this section, we shall interpret these flows in terms of *social mobility*.

A first distinction has to be made at this stage. Different mobility structures may, first of all, determine different equilibrium class distributions: this is what sociologists label as *structural mobility*. This notion refers to the idea that, via the equilibrium distribution they produce, different mobility structures imply different availability of positions in higher or lower social classes. This is not, however, the only way to look at mobility: different mobility structures also influence the intertemporal movement of individuals among the social classes, for a given equilibrium distribution. This latter effect, known as *exchange* (or *pure*) mobility can be regarded as the dynamic counterpart of the comparative statics on different income distributions characterized by the same average income. It is this effect which we examine in this paper.

First some notation. Let $q(i'|i)$ denote the (not necessarily equilibrium) transition probability for an individual initially in class i to move to class i' after participation in a game. If the population is using a custom, k, and the prior state of the system is $\mathbf{x} \in \Delta^4$, then it follows from (3.1a) that

$$q(i'|i) = \sum_{j',j} p(i',j'|i,j) x_j . \qquad (5.1)$$

Note that $q(i'|i)$ is independent of whether the player is player-I or player-II, and is the same in either case by the symmetry condition (3.1b). Using the formulae in equations (4.1), we therefore obtain

$$q(i'|i) = \frac{1}{2} x_i + \sum_{j \neq i} [k(j,i)\delta^{j}_{i'} + k(i,j)\delta^{i}_{i'}] x_j . \qquad (5.2)$$

Denote by $Q(k)$ the (state dependent) matrix of transition probabilities, $q(i'|i)$, with i labelling rows and i' columns. Then, for example, the transition matrices for customs k_1 and k_{16} (that is, the two extremes in the spectrum of customs shown in Table 2.1):

$$Q(1) = \begin{pmatrix} \frac{1}{4}x_1 & 1 - \frac{3}{4}x_1 & \frac{1}{4}x_1 & \frac{1}{4}x_1 \\ \frac{1}{4}x_2 & 1 - x_1 - \frac{3}{4}x_2 & \frac{1}{4}x_2 & x_1 + \frac{1}{4}x_2 \\ \frac{1}{4}x_3 & \frac{1}{4}x_3 + x_4 & \frac{1}{4}x_3 & 1 - \frac{3}{4}x_3 - x_4 \\ \frac{1}{4}x_4 & \frac{1}{4}x_4 & \frac{1}{4}x_4 & 1 - \frac{3}{4}x_4 \end{pmatrix}, \tag{5.3a}$$

$$Q(64) = \begin{pmatrix} \frac{1}{4}x_1 & \frac{1}{4}x_1 & \frac{1}{4}x_1 & 1 - \frac{3}{4}x_1 \\ \frac{1}{4}x_2 & x_1 + \frac{1}{4}x_2 & \frac{1}{4}x_2 & 1 - x_1 - \frac{3}{4}x_2 \\ \frac{1}{4}x_3 & 1 - \frac{3}{4}x_3 - x_4 & \frac{1}{4}x_3 & \frac{1}{4}x_3 + x_4 \\ \frac{1}{4}x_4 & 1 - \frac{3}{4}x_4 & \frac{1}{4}x_4 & \frac{1}{4}x_4 \end{pmatrix}. \tag{5.3b}$$

Noticethat Columns 1 and 3 areequal, and that $Q(64)$ is obtained from $Q(1)$ by interchanging columns 2 and 4. Of course, $Q(k)$ is the (conditional) transition matrix which characterises the stochastic process faced by a single player in a society operating custom k, conditional on the class distribution being $\mathbf{x}$. From this *individual* perspective, x_i must be interpreted as the probability that an individual chosen at random from the population, belongs to class i. We call this the *individual* process. Note that these transition probabilities are the same for each player belonging to the same class. If we consider these probabilities as *proportions* , the same transition matrix $Q(k)$ expresses, at a *population* level, the expected motion of a population in which the proportion in class i is represented by x_i[14]. The (expected) class distribution in the next time period will then be:

$$\mathbf{x}(t+1) = \mathbf{x}(t)Q(k). \tag{5.4}$$

A caveat here. In analysing the mobility structure of our society, we will make constant reference to these transition matrixes $Q(k)$, as is customary practise in the literature on social mobility. The transition matrix $Q(k)$ desribes a well-defined stochastic process over the set of classes which is *not*, however, the one described in section 2. In particular, if we desribe the transition from class to class by means of the indivudal process $Q(k)$, we notwithstand the fact that the actual transition process comes as a result of *a game* being played between two agents randomly paired (as a matter of fact, we have always considered transition probabilities of the form $(i, j) \to (i', j')$). Nonetheless, given that $Q(k)$ is, by construction, the transition matrix which characterises the stochastic process faced by a single player *before* she has been paired,

[14] See Kemeny and Snell [1976], sec. 6.

and there is no correlation between the random matching and the (possibly mixed) strategy profile that may be played, it would be surprising if the population process described by $Q(k)$ produced a different limiting class distribution than the one we constructed in the previous section. Therefore, for consistency with the development in section 4, we expect (and obtain) the following:

PROPOSITION 5.1. For the equilibrium distribution, $\bar{\mathbf{x}}$, given by Proposition 4.2, we have $\bar{\mathbf{x}} = \bar{\mathbf{x}}\bar{Q}(k)$, where $\bar{Q}(k)$ is the transition matrix at $\bar{\mathbf{x}}$. That is, $\bar{\mathbf{x}}$ is an equilibrium of the (non-linear) discrete-time dynamic (5.4). In fact, $\bar{\mathbf{x}}$ is the unique global attractor for this dynamic.

Proof. See Appendix A. □

We now move to social mobility, and compare the universal equilibrium $\bar{\mathbf{x}}$ with respect to its exchange structure under different customs. Following a well established tradition, we do this with the aid of a *mobility index*. Among the various alternative indices proposed in the literature, we choose the Bartholomew [1973] index, defined as follows:

Definition 5.1. The *Bartholomew mobility index* for custom k is defined by

$$B(k) = \sum_{i,i'} |i - i'| \bar{q}(i'|i) \bar{x}_i \qquad (5.5)$$

where $\bar{q}(i'|i)$ is the (i, i')-th entry in the equilibrium transition matrix $\bar{Q}(k)$.

Thisindexis, of course, justtheexpectedvalueinequilibrium ofthepossible(non-directional) class changes, $|i - i'|$, representing the possible changes in individual class resulting from a play of the game. The mobility index is easily calculated for each of the 64 possible customs, and we obtain:

PROPOSITION 5.2. The mobility index B induces a linear ordering on the set of customs. This ordering identifies k_1 as having the minimum mobility, and k_{64} as having the maximum mobility.

Proof. See Appendix A. □

The intuition behind the result is not difficult to understand. What follows is the transition matrix $\hat{Q}(1)$ evaluated when the society is at equilibrium, $\bar{\mathbf{x}}$, given by Proposition 4.2:

$$\hat{Q}(1) = \begin{pmatrix} 0.0221405 & 0.933578 & 0.0221405 & 0.0221405 \\ 0.102859 & 0.602859 & 0.102859 & 0.191422 \\ 0.0221405 & 0.433578 & 0.0221405 & 0.522141 \\ 0.102859 & 0.102859 & 0.102859 & 0.691422 \end{pmatrix}, \qquad (5.6)$$

Note that $\hat{q}(4|4) = 0.69$ and $\hat{q}(2|2) = 0.6$. This means that an agent who belongs to one of the most represented classes (in equilibrium) at time t, will stay in the same class at time $t+1$ with a fairly high probability. The reason is that a player of class 2 under custom k_1 always cooperates against higher-class opponents (while a player of class 4 will defect in return). Therefore, after the encounter, each will find herself in the same class as before the play. Consider instead what happens under custom k_{64}:

$$\hat{Q}(64) = \begin{pmatrix} 0.0221405 & 0.0221405 & 0.0221405 & 0.933578 \\ 0.102859 & 0.191422 & 0.102859 & 0.602859 \\ 0.0221405 & 0.522141 & 0.0221405 & 0.433578 \\ 0.102859 & 0.691422 & 0.102859 & 0.102859 \end{pmatrix}. \qquad (5.7)$$

Now $\hat{q}(4|4)$ has gone down to 0.1 and $\hat{q}(2|2) = 0.19$ (while $\hat{q}(2|4)$ and $\hat{q}(4|2)$ have moved up from 0.19 to 0.6 and from 0.1 to 0.69 respectively). This is because, under custom k_{64}, the social ranking is always reversed after the play, enhancing the overall mobility of the society.

It might be worth noting that the intuitive appeal of Proposition 5.2 is not to be taken for granted. Alternative indices do not produce the same clear-cut result, as is well known in the literature which focuses on social mobility [14]. The results of this section specify, in a formal way, why we think of k_{64} (i.e. a code of behaviour which consistently favors the lower class player in the division of the pie) as a code of conduct which promotes social mobility, and custom k_1 (where the opposite holds) as a code which discourages it. For this reason, we will hereafter refer to k_1 as the *Immobile Custom* and to k_{64} as the *Mobile Custom*.

6. Two-custom society

[14] See, for example, Dardanoni [1993].

Up to this point, we have studied the case of a society which unanimously agrees on a unique custom. We now move to a setting in which we allow the possibility of an heterogeneous society. In this section we analyse the simplest possible case, in which agents use one of two possible customs. In the following section the analysis will be extended to a society in which all 64 customs may be present.

Different codes of behaviour are followed by different players who, occasionally, interact. The first, intuitive, implication is that coordination on one of the possible Nash equilibria of the Class Game is no longer guaranteed when agents belonging to different classes meet.[16] It may happen that both customs prescribe the same pure strategy; so that people fail to play optimally. If guaranteeing an optimal play is what a custom is for, the simple coexistence of multiple customs creates, within the constraints of our simple model, a clear inefficiency, due to the fact that players now miscoordinate much more often. The extent to which this problem can arise depends, of course, upon the relative frequences of the different customs. Think, for example, of a custom which is comparatively rare: people who follow it are more likely to mismatch their behaviour compared to those who follow ”more popular” customs (this is, essentially, because the ”rare” custom fails to act as a coordinating device). If so, it is reasonable to assume that, when multiple customs coexist, some kind of *coordination learning* might take place in the population; i.e. agents modify their custom in the light of experience, with a view to finding better coordination devices.

In addition, we consider a further source of learning, which we label *aspiration learning*. According to this, an agent will change her custom with positive probability only if her realised payoff lies *below* a threshold value, which partially depends on her class (and is therefore *endogenous*), and partly on some *exogenous* constant, which is fixed and common for all the individuals in the population. In both cases, we shall assume that the probability with which an agent may switch her custom will be *proportional* to the difference between these threshold values and the game payoffs, as will be specified explicitely shortly.

[16] Remember that when two players belonging to the same class meet, any custom prescribes the same mixed stratey (i.e. $k(i, i) = 1/2$, for any $k \in K$).

We now give a formal description of this two-custom society. There are two customs, k_A and k_B. A player's (instantaneous) state is represented by a pair (i,α), with $i \in \{1,2,3,4\}$ and $\alpha \in \{A,B\}$. We denote by $\bar{\alpha}$ the complementary custom to α in $\{A,B\}$. Thus, $\bar{A} = B$ and $\bar{B} = A$.

If player-I has state (i,α) and player-II has state (j,β), then the game results in state transitions, $(i,\alpha) \to (i',\alpha')$ and $(j,\beta) \to (j',\beta')$. This transition occurs with probability

$$p(i',j';\alpha',\beta'|i,j;\alpha,\beta). \tag{6.1}$$

Our first aim is to compute all the possible non-zero transition probabilities (6.1).

First note that the possible transitions are not completely determined by the game alone, because now each player may change her custom by applying either a *coordination test*, with probability $(1-\lambda)$, or an *aspirationtest*, with probability λ, where $\lambda \in (0,1)$ is some exogenous constant, *after* the Class Game has been played:

The coordination test. We say that the customs used by the two players *coordinate at* (i,j) if

$$k_\alpha(i,j) = k_\beta(i,j). \tag{6.2}$$

The customs therefore *fail* to coordinate at (i,j) if $k_\alpha(i,j) \neq k_\beta(i,j)$. Of course, when there are only two possible customs, this can only happen if $\beta = \bar{\alpha}$. Note also that, if $i = j$ the two customs always coordinate at (i,j).

When player-I applies a coordination test, then $\alpha' = \alpha$ if α and β coordinate at (i,j), and $\alpha' = \bar{\alpha}$ otherwise. Note that a failure to coordinate is detected by player-I from her subsequent class i'. The public information available to both players prior to the game is the pair of class numbers (i,j); informationabout theotherplayer's custom is notavailable. Thus, ifthecustoms coordinate at (i,j) with $i = j$, then $(i',j') = (2,4)$ if $k_\alpha(i,j) = 0$ (player-I Cooperates), and $(i',j') = (4,2)$ if $k_\alpha(i,j) = 1$ (player-I Defects). However, if there is a failure of coordination, then $(i',j') = (3,3)$ if $k_\alpha(i,j) = 0$, and $(i',j') = (1,1)$ if $k_\alpha(i,j) = 1$.

The intuition is the following. If the main function of a custom is to lead to coordination, this is what one should check first. In this respect, we should expect each custom to work as well as any other. We confine our attention to pure strategy outcomes for the following reason. As already noticed, a custom operates effectively only when two players from different classes meet, exactly the situation where a player would expect to be guided by the custom toward an optimal play. If this does not happen, then it is reasonable to assume that players may cast doubt on the validity of the custom they follow. We interpret this process as taking place on an individual level. Thus, there is a positive probability $(1 - \lambda)$, which we assume to be the same for each player, that, after the game has been played, each agent applies a test of this kind, and updates her custom accordingly. As mentioned previously, with the remaining probability λ, each agent will judge the performance of her custom from a different perspective, as follows.

The aspiration test. If player-I applies an aspiration test, then she will change her custom with a probability $\gamma_{ii'}$, which depends only on her class change $i \to i'$. The particular form of the aspiration test probabilities we shall consider is an amalgam of two complementary tests, whose relative weight is measured by an exogenous constant $\eta \in (0, 1)$, which is assumed to be the same for all individuals in the population. These are defined as follows:

The endogenous aspirationtest. Under this test, each individual compares her relative position before and after the encounter. We assume that this part of the test will lead to a change in the custom only when there is a status loss, i.e. when $(i - i') > 0$, and that the probability of such a change is proportional to this loss.

The exogenous aspiration test. Under this test, each individual compares the game outcome with some exogenous constant $\sigma \in [1, 4]$, here meant to represent a commonly shared "social standard" of what should be considered a *fair* split of the cake. Given that this comparison could be performed equally with the prior class i, or with the posterior class i', and we have no definite criterion for preferring one over the other (given that each alternative has its pros and cons), we assume that each individual will average out the class transition, comparing the social standard σ with $\frac{(i+i')}{2}$. As for the case of the endogenous aspiration test, we assume that this part of the test will lead to a change in the custom only when a player's averaged position is still below what is considered socially fair, i.e. when $(\sigma - \frac{(i+i')}{2}) > 0$. Moreover, we will assume that this probability also will be proportional to the difference $(\sigma - \frac{(i+i')}{2})$. The exact form in which the two parts of the aspiration test, exogenous and endogenous, are combined together to determine the transition probabilities $\gamma_{ii'}$ is given by

$$\gamma_{ii'} = \frac{1}{3}\eta\delta[\sigma - \frac{i + i'}{2}] + \frac{1}{3}(1 - \eta)\delta[i - i'], \qquad (6.3)$$

where $\eta \in (0, 1)$ measures the relative weight of the exogenous aspiration test and $\delta[x] = x$ if $x > 0$, $\delta[x] = 0$ otherwise. [The factor $\frac{1}{3}$ is for normalization purposes].

We provide a justification for this (rather peculiar) structure of the updating process, which is driven by two (rather different) forces. While coordination learning implicitely assumes that the agents are well aware of the fact that the outcome of the Class Game is the product of some interactivedecision,thosewhoupdatetheircustomaccordingtotheaspirationtestneednotknow anything about the strategic features of situation in which they are involved (apart from the share of the pie they obtain). Otherwise, they simply do not care. While coordination learning recalls the classic ”best-reply dynamics” over the space of customs (given that such a learning protocol is active only if an agent has not played a best response against the opponent’s move), with our aspiration test we try to model some form of ”learning through reinforcement”, the object of recent interest both in the learning and experimental literature. These are the two learning models which have been given most attention by economists, as both learning schemes seem to provide a suitable framework in the economic modelling of boundedly rational agents, and their predictions have both empirical and experimental support. [17] While coordination learning suits environments in which the agents play strategically (though not in a very sophisticated way), our aspiration learning seems to be more appropriate in situations in which people know or care very little of the strategic aspects of the environment in which they act. We do not have, in

[17] Herewe considerbest-reply dynamicsas aspecial case of abroaderclass of adjustment process, namely *adaptive learning dynamics*, following the terminology of Milgrom and Roberts [1991]. Learning procedures similar to our aspiration test have been studied recently by, among others, Bendor et al. [1991] and Börgers and Sarin[1994]. For the experimental evidence, see Mookerjee and Sopher [1994] and Roth and Erev [1983]. Proportional learning rules have been proposed by Cabrales [1993], and Schlag [1994], who also provides conditions under which a similar adjustment process can be justified on normative grounds.

principle, any reason to favor one learning protocol over the other, and we are actually interested in testing the predictions of our model in the presence of both these effects, using λ, that is, the probability with which an agent will choose one test or the other, as a control variable in our simulations. As people hold different customs, they can also react in different ways, and we allow some degree of freedom in modelling the updating process, appealing to the two most influencial candidates learning theory has provided so far.

To summarise, in a two-custom society, if player-I applies an aspiration test (with probability λ), then $a' = a$ in (6.1) with probability $(1 - \gamma_{cc'})$ (i.e. there is no change of custom), and $a' = \bar{a}$ with probability $\gamma_{cc'}$ (i.e. there is a change of custom).

In appendix B we compute the transition probabilities (6.1) for general $\gamma_{cc'}$. We also prove the following.

PROPOSITION 6.1. For $0 < \lambda, \eta < 1$ and $N \geq 2 \times 4 = 8$, the two-custom Markov process defined on the lattice $\Delta_N^{(N_1,N_2)} \subset \Delta^7$ is ergodic.

Proof. See Appendix D. □

For custom A, let $\mathbf{x}_A = (x_{(1,A)}, x_{(2,A)}, x_{(3,A)}, x_{(4,A)})$ be the vector of proportions in each of the four classes, and let $\mathbf{x}_B = (x_{(1,B)}, x_{(2,B)}, x_{(3,B)}, x_{(4,B)})$ be the vector of proportions for custom B. Thus, the total vector, $\mathbf{x} = (\mathbf{x}_A, \mathbf{x}_B) \in \Delta^7 \subset \mathbb{R}^8$. The deterministic equations (3.6) now have the form

$$\frac{dx_{(k,\rho)}}{dt} = \sum_{(i,\alpha)} \sum_{(j,\beta)} c^{(i,\alpha),(j,\beta)}_{(k,\rho)} x_{(i,\alpha)} x_{(j,\beta)} \qquad (6.4)$$

where the coefficients (3.5) are given by

$$c^{(i,\alpha),(j,\beta)}_{(k,\rho)} = \sum_{(l,\sigma)} \left\{ p(k,l;\rho,\sigma|i,j;\alpha,\beta) + p(l,k;\sigma,\rho|i,j;\alpha,\beta) \right\} - \left\{ \delta^{(i,\alpha)}_{(k,\rho)} + \delta^{(j,\beta)}_{(k,\rho)} \right\} \qquad (6.5)$$

Substituting from (6.5) into (6.4) and using the symmetry condition (3.1b), we obtain

$$\frac{dx_{(k,\rho)}}{dt} = -2x_{(k,\rho)} + 2 \sum_{(i,\alpha)} \sum_{(j,\beta)} \left\{ \sum_{(l,\sigma)} p(k,l;\rho,\sigma|i,j;\alpha,\beta) \right\} x_{(i,\alpha)} x_{(j,\beta)} \qquad (6.6)$$

An explicit form for these equations is computed in Appendix B. We cannot provide a formal analysis of the solutions of the system (6.6), whose properties will be derived by simulation. Nonetheless, before we proceed, it may be interesting to analyse how the learning protocols we have designed would operate if they were applied to the exchange structure defined by $\hat{Q}(k)$ considered in the previous section. In other words, as an excercise, we try to gain intuition about the selection process over the custom space looking at the probabilities with which, given that a one-custom society has reached the equilibrium distribution $\bar{\mathbf{x}}$, our coordination and aspiration tests *would* lead to an individual changing her custom.

It is already obvious that such an excercise can be carried out only for the aspiration test, since our coordination test, by construction, will never fail when everybody follows the same custom. In what follows, we will therefore calculate the effects of the various parts of the aspiration test if it were to be applied in a one-custom society. This should be seen as a measure of the ease with which a one-custom society at equilibrium could be invaded by mutants who apply an aspiration test with some probability. This is therefore a measure of *social stability*.

Suppose we are given a matrix of real numbers, $a = \{a_{ii'} | 1 \le i, i' \le n\}$. Then we can define an a*-mobility index* on the set of customs, by

$$I_a(k) = \sum_{i,i'} a_{ii'} \hat{q}(i'|i) \hat{x}_i \tag{6.7}$$

where, as usual, the 'hat' refers to evaluation at the equilibrium $\bar{\mathbf{x}}$. An a-mobility index of the form (1) induces an ordering on the set of customs by:

$$k_1 \le_a k_2 \quad \text{if and only if} \quad I_a(k_1) \le I_a(k_2). \tag{6.8}$$

We shall be concerned with ordering customs according to various indices of this type.

If we think of $a_{ii'}$ as a 'reward' (if $a_{ii'} > 0$), or a 'penalty' (if $a_{ii'} < 0$), payable on an agent's transition from class i to class i' after playing the stage game, then $I_a(k)$ is just the expected reward, at equilibrium, when everyone uses the custom k. A custom which has a high a-mobility index, therefore, has a high expected a-reward. For example, when $a_{ii'} = |i - i'|$, then $I_a(k) = B(k)$ is just the Bartholomew mobility index, and a custom with a high 'expected reward' corresponds to a more mobile society. In this section, we shall be mainly interested in the case in which the $a_{cc'} = \gamma_{cc'}$, the aspiration test probabilities for a custom change. Thus, $\bar{I}_\gamma(k)$ is just the expected probability, at equilibrium, that an application of the aspiration test will lead to a change of custom.

Now recall that $\gamma_{cc'} = \eta\gamma^{ex}_{cc'} + (1-\eta)\gamma^{en}_{cc'}$, splits into two components, an exogenous part, $\gamma^{ex}_{cc'} = \frac{1}{2}k[\sigma - \frac{c+c'}{2}]$, and an endogenous part, $\gamma^{en}_{cc'} = \frac{1}{2}k[c - c']$. Hence, we may write,

$$\bar{I}_\gamma(k) = \eta\bar{I}_{ex}(k) + (1-\eta)\bar{I}_{en}(k), \tag{6.9}$$

where $\bar{I}_{ex}(k)$ is the γ^{ex}-mobility index, and $\bar{I}_{en}(k)$ is the γ^{en}-mobility index. Clearly, $\eta\bar{I}_{ex}(k)$ is just the expected probability that the exogenous part of the aspiration test will lead to a change of custom, and similarly, $(1-\eta)\bar{I}_{en}(k)$, is the expected probability for the endogenous part of the test. We aim to prove:

PROPOSITION 6.2. *(i)* $\bar{I}_{en}(\cdot)$ induces the same ordering on the set of customs as the Bartholomew index.

(ii) When $\sigma = 3$, $\bar{I}_{ex}(\cdot)$ induces the reverse of the Bartholomew index ordering on the set of customs.

(iii) When $\sigma = 4$, $\bar{I}_{ex}(k)$ is independent of k, and so induces the uniform ordering on the set of customs.

Before proving the proposition, we first need some lemmas.

LEMMA 6.1. For a matrix a, define an index $\tilde{I}_a(k)$, by

$$\tilde{I}_a(k) = \sum_{c,j} (a_{j2} - a_{j1})k(c,j)\pi_c\pi_j. \tag{6.10}$$

Then $\tilde{I}_a(\cdot)$ induces the same ordering on the set of customs as $\bar{I}_a(\cdot)$.

Proof. Recall that $p(c'|c) = \frac{1}{2}\pi_c + \sum_{j\neq c}|k(j,c)\delta^{?}_{c} + k(c,j)\delta^{?}_{c}|\pi_j = \frac{1}{2}(1 - \delta^{?}_{c} - \delta^{?}_{c})\pi_c +$

$\sum_j |k(j,i)\delta_i^2 + k(i,j)\delta_i^1| w_j$. Thus,

$$\begin{aligned} \tilde{I}_\alpha(k) &= \frac{1}{4}\sum_{i,i'} \alpha_{ii'} \bar{w}_i^2 - \frac{1}{4}\sum_i (\alpha_{i2} + \alpha_{i1}) \bar{w}_i^2 + \sum_{j,i} |\alpha_{i2} k(j,i) + \alpha_{i,1} k(i,j)| \bar{w}_j \bar{w}_i \\ &= \left\{ \frac{1}{4}\sum_i (\alpha_{i1} + \alpha_{i2}) \bar{w}_i^2 + \sum_i \alpha_{i1} \bar{w}_i \right\} + \sum_{j,i} (\alpha_{i2} - \alpha_{i1}) k(j,i) \bar{w}_j \bar{w}_i \\ &= \tilde{A}_\alpha + \tilde{J}_\alpha(k), \end{aligned}$$

where $\tilde{A}_\alpha$ is independent of the custom k. It therefore follows from the definition (2) (and the corresponding definition for $\tilde{J}_\alpha$), that $\tilde{I}_\alpha$ and $\tilde{J}_\alpha$ induce the same ordering on the set of customs.

□

LEMMA 6.2. For any custom k, $\sum_{i,j} k(i,j) w_i w_j = \frac{1}{2}$.

Proof. $\sum_{i,j} k(i,j) w_i w_j = \sum_{i,j} [1 - k(j,i)] w_i w_j = 1 - \sum_{i,j} k(j,i) w_i w_j$. Now interchange dummy indices in the right hand sum to obtain the result. □

LEMMA 6.3. Let $\alpha = \{\alpha_{ii'}\}$ and $\beta = \{\beta_{ii'}\}$ be matrixes. Suppose there are real numbers, a and b, such that $a(\alpha_{i2} - \alpha_{i1}) + b(\beta_{i2} - \beta_{i1}) = 1$ for each i. Then $\tilde{I}_\alpha(\cdot)$ and $\tilde{I}_\beta(\cdot)$ induce the same (resp. reverse) ordering on the set of customs if $ab < 0$ (resp. $ab > 0$).

Proof. By Lemma 6.1, it suffices to prove the result for $\tilde{J}_\alpha(\cdot)$ and $\tilde{J}_\beta(\cdot)$. But, $a(\alpha_{i2} - \alpha_{i1}) + b(\beta_{i2} - \beta_{i1}) = 1$, together with Lemma 6.2, implies that

$$a\tilde{J}_\alpha(k) + b\tilde{J}_\beta(k) = \frac{1}{2}.$$

Suppose $ab = 0$. Then $k_1 \leq_\alpha k_2$ if and only if $\tilde{J}_\alpha(k_2) - \tilde{J}_\alpha(k_1) \geq 0$; if and only if $a^2 [\tilde{J}_\alpha(k_2) - \tilde{J}_\alpha(k_1)] \geq 0$; if and only if $ab[\tilde{J}_\beta(k_2) - \tilde{J}_\beta(k_1)] \leq 0$; if and only if, either $ab < 0$ and $k_1 \leq_\beta k_2$, or $ab > 0$ and $k_1 \geq_\beta k_2$. □

Proof of Proposition 6.2. *(i).* Let $\alpha_{ii'} = |i - i'|$, so that $\tilde{I}_\alpha(k) = B(k)$ is the Bartholomew index, and $\beta_{ii'} = 3\gamma_{ii'}^{**} = h[i - i']$. Then,

$$\begin{aligned} \{\alpha_{i2} - \alpha_{i1}\} &= \{-2, -2, 0, 2\}, \\ \{\beta_{i2} - \beta_{i1}\} &= \{0, 0, 1, 2\}. \end{aligned}$$

Hence, $-\frac{1}{2}(\alpha_{i2} - \alpha_{i4}) + (\beta_{i2} - \beta_{i4}) = 1$ for each i. Since $-\frac{1}{2} \times 1 = -\frac{1}{2} < 0$, it follows from Lemma 6.3 that α and β induce the same ordering on the set of customs. This proves *(i)*.

(ii). Let $\alpha_{ii'} = |i - i'|$, as above, and $\beta_{ii'} = 3\gamma^{**}_{ii'} = h[3 - \frac{i+i'}{2}]$. Then,

$$\{\beta_{i2} - \beta_{i4}\} = \{1, 1, \frac{1}{2}, 0\}.$$

Hence, $\frac{1}{2}(\alpha_{i2} - \alpha_{i4}) + 2(\beta_{i2} - \beta_{i4}) = 1$ for each i. Since $\frac{1}{2} \times 2 = 1 > 0$, it follows from Lemma 6.3 that α and β induce the reverse ordering on the set of customs. This proves *(ii)*.

(iii). When $\sigma = 4$, set $\beta_{ii'} = 3\gamma^{**}_{ii'} = (4 - \frac{i+i'}{2})$. Then, $\{\beta_{i2} - \beta_{i4}\} = \{1, 1, 1, 1\}$. Thus, $\bar{J}_\beta(k) = \frac{1}{2}$ by Lemma 6.2, which is independent of the custom k. It follows that $\bar{J}_\beta(\cdot)$, and hence $\bar{J}_{**}(\cdot)$, induces the uniform ordering on the set of customs. This proves *(iii)*. □

For a given matrix α, the α-ordering is determined by the J_α-index (6.10). We therefore attempt to give an interpretation of this index.

First note that, given that player-II has class j, $\bar{P}_j(D) = \sum_i k(i,j)w_i$ is the probability (at equilibrium) that player-I will Defect. If, on this event, and after having played his own strategy (dictated by $k(i,j)$), player-II moves into class 2, then he receives a 'reward' α_{j2}. On the other hand, if he moves into class 4 he receives a reward α_{j4}. If he does 'better' (in terms of the α-reward scheme) in the former case, then $\alpha_{j2} > \alpha_{j4}$, whereas the reverse is true if he does better in the latter case. Thus, if moving into class 2 is more advantageous to player-II, as measured by the reward scheme α, than moving into class 4, then it is advantageous to player-II that player-I should Defect (*i.e.* $\bar{P}_j(D)$ should be high). Conversely, if it is more advantageous to move into class 4, then it is advantageous to player-II that player-I should Cooperate.

In terms of the reward scheme for the endogenous part of the aspiration learning rule, $\alpha_{ii'} = 3\gamma^{**}_{ii'} = h[i - i']$, we have

$$\{\alpha_{j2}\} = \{0, 0, 1, 2\}, \qquad \{\alpha_{j4}\} = \{0, 0, 0, 0\}.$$

Thus, for $j = 3, 4$ it is advantageous for player-II that player-I should Defect, and for $j = 1, 2$, player-II is indifferent to player-I's strategy. Thus, $\bar{P}_3(D)$ and $\bar{P}_4(D)$ should be high.

The former has its maximum when $k(1,3) = k(2,3) = k(4,3) = 1$, and the latter when $k(1,4) = k(2,4) = k(3,4) = 1$. These requirements are incompatible at $(3,4)$. However, the relevant terms in $\bar{I}_{\sigma}(k) = \sum_j (a_{j3} - a_{j4}) w_j P_j(D)$, are $w_3 k(4,3) w_4$ and $2 w_4 k(3,4) w_3$. Thus, since $2 w_4 w_3 > w_3 w_4$ it is more advantageous for the overall index that $k(3,4) = 1$ rather than $k(4,3) = 1$. If we write

$$\kappa(k) = \{k(1,2), k(1,3), k(1,4), k(2,3), k(2,4), k(3,4)\},$$

we therefore find that customs with the highest $\bar{I}_{41}(k)$-mobility index, satisfy

$$\kappa(k) = \{*, 1, 1, 1, 1, 1\}, \qquad (6.11)$$

where $*$ can be either 0 or 1. In particular, the Mobile Custom, $\kappa(k_{64}) = \{1,1,1,1,1,1\}$, has the highest possible $\bar{I}_{41}(k)$-mobility index, as does $\kappa(k_{32}) = \{0,1,1,1,1,1\}$. A similar analysis shows that the Immobile custom, $\kappa(k_1) = \{0,0,0,0,0,0\}$, has the lowest possible $\bar{I}_{41}(k)$-mobility index, as does $\kappa(k_{33}) = \{1,0,0,0,0,0\}$.

We can play the same game with the exogenous part of the aspiration test learning rule, $a_{ii'} = 3\gamma^{44}_{ii'} = h[3 - \frac{i+i'}{2}]$, when $\sigma = 3$. In this case,

$$\{a_{j3}\} = \{\frac{3}{2}, 1, \frac{1}{2}, 0\}, \qquad \{a_{j4}\} = \{\frac{1}{2}, 0, 0, 0\}.$$

Thus, to ensure a high expectation of custom change from the exogenous part of the aspiration test, we require $P_j(D)$ to be large for $j = 1,2,3$, which in turn requires that $k(2,1) = k(3,1) = k(4,1) = 1$, and $k(1,2) = k(3,2) = k(4,2) = 1$, and $k(1,3) = k(2,3) = k(4,3) = 1$. The incompatibilities here occur at $(1,2), (1,3), (2,3)$, and the relevant choices are between the pairs of terms in the I_{σ}-index, $\{w_2 k(1,2) w_1, w_1 k(2,1) w_2\}$, $\{\frac{1}{2} w_3 k(1,3) w_1, w_1 k(3,1) w_3\}$ and $\{\frac{1}{2} w_3 k(2,3) w_2, w_2 k(3,2) w_3\}$. For the first pair, it is a matter of indifference whether we take $k(1,2) = 1$ or $k(2,1) = 1$; for the second pair, it is more advantageous to take $k(3,1) = 1$; and for the third pair, it is more advantageous to take $k(3,2) = 1$. Thus, to achieve a maximum of $\bar{I}_{44}(k)$, we require $\kappa(k)$ to have the form,

$$\kappa(k) = \{*, 0, 0, 0, 0, 0\}.$$

In particular, this occurs when $k = k_1$. Conversely, $I_{ss}(k)$ is a minimum when $n(k) = \{4,1,1,1,1,1\}$; *e.g.* when $k = k_{64}$.

Thecontent of Proposition6.2 can berephrased asfollows. Ifpeople mainly look attheir class change to determine whether to keep their custom or not (Proposition 6.2(i)), then less mobile customs exhibit strongerstability properties (inthe sensethat, giventheyare alreadyestablished, they minimize the probability of a failure in the aspiration test). On the other hand, if people mainly care about fairness considerations (Proposition 6.2(ii)), the opposite will occur, and we can expect more mobile customs to be predominant.[14] We test our (preliminary) conclusions by simulations, evaluating numerical solutions of the system (6.6). It seems somehow natural to start looking at the case in which people follow the two ”extreme” customs, i.e. when $k_A = 1$ and $k_B = 64$. In this case, the society is split into two subgroups which follow, respectively, the Immobile and the Mobile Custom.

[14] Even ifthe”social standard” σ couldtake, inprinciple, any value within theinterval $[1,4]$, werestricted our analysis to the cases $\sigma = 3$ and $\sigma = 4$ for the following reasons. The choice of $\sigma = 4$ describes a situation in which an individual aims to reach the top of the social ranking, regardless of her current status (since 4 is the maximum payoff she can achieve in the Class Game). The choice $\sigma = 3$ can be justified on fairness grounds, since, when $\sigma = 3$, the utility pie is equally divided.

FIGURE 6.1. *Mobile vs. Immobile Custom.*

The diagrams of Figure 6.1 show the limiting class distributions under both customs, as well as population shares and average payoffs under four different configurations of the parameter pair $\{\lambda, \eta\}$.[19] In the last column of the matrix associated with each diagram, the Bartholomew indexes for the one-custom case are to be compared with the one exhibited by the (equilibrium) two-custom society.

First notice that the population share of those who follow the Imobile Custom *never* exceeds $1/2$, while it is substantially smaller than this value when η is high (i.e. when the exogenous aspiration test is performed with suficiently high probability). When λ is also high (i.e. when the aspiration test is applied much more frequently than the coordination test) the proportion of the population which follows the Mobile Custom is almost as twice as much as the proportion which follows the Immobile Custom.

While this latter result is consistent with the content of Proposition $6.2(ii)$ (since the conjunction of λ and η high, with $\sigma = 3$ provides in principle the "best of the possible environments" for the Mobile Custom), the overall poor performance (in terms of limiting population share) of the Immobile Custom under any parameter configuration is puzzling. One reason for this outcome might be the somehow arbitrary choice of the two contestants, placed at the opposite estremes of the custom space. To test this, we stage a Round Robin tournament among all the customs in $\mathcal{C}$, in which each custom is paired against each of the other 63 to form a two-custom society. Table 6.1 summarises the relevant summary statistics of the performance of the Mobile and Immobile customs in the tournament described above, under the same parameter settings as the previous example:

TABLE 6.1. *A Round Robin tournament.*

Once again, while it seems to be true that an environment characterised by both λ and η small favors the Immobile Custom[20], it is also true that the Mobile Custom obtains a (much) larger

[19] In the simulation displayed in Figures 6.1-2, we fixed $\sigma = 3$, although (somewhat surprisingly) this choice does not seem to affect substantially the essence of the results (see Table 6.1 below).

[20] Notice, however, that when λ and η are both small, the relative performance of the various customs tends to converge, as in the simulations shown in Figure 6.1

share of the population in all the other cases, especially in the case when $\{\lambda, \mu\} = \{.999, .001\}$, where we should still expect a good performance of the Immobile Custom against any possible contestant (given that μ is relatively "small").

Our tournament suggests that there is something missing if we look at the data relying only on the conclusions of Proposition 6.2. Moreover, intuition suggests that the missing factor is to be found in the effects of the aspiration test, since, in a two-custom society, whenever coordination does not take place, there is an equal push against both customs (and therefore these opposite pushes should in principle cancel out).

To proceed with the analysis, we display a detailed summary of the encounter between the two least mobile customs, i.e. k_1 and k_{33}:

FIGURE 6.2. *The two least mobile customs (1 vs. 33)*

Once again, the limiting population share of the Immobile Custom never exceeds $1/2$, with the gap increasing with μ and λ, as it happened in k_1 *vs.* k_{64} case.

Remember that both customs, k_1 and k_{33} exhibit the same mobility in the one-custom society, at least when mobility is measured with the aid of the equilibrium Bartholomew index. In other words, one cannot appeal to mobility alone to explain why our dynamic seems to work against the Immobile Custom, as this also happens when k_1 is paired with a custom which exhibits the same mobility.

Note that the only difference between k_1 and k_{33} is that $k_1(1,2) = 0$ while $k_{33}(1,2) = 1$. In otherwords, k_{33} prescribes thesamebehaviour asthe Immobile Custom, under all contingencies except when an individual of class 1 meets an opponent of class 2.

We shall look at this encounter in more detail. To do so, some further terminology is needed. For $a \in \{A, B\}$, let $\phi_{ij}(a, x)$ denote the conditional probability that player-I changes her custom by application of the aspiration test, given that (i) she has class i, (ii) her prior custom is k_a, (iii) player-II has class j, and (iv) the (not necessarily equilibrium) state of society is

$\mathbf{x} = \left(x_{(1,A)}, x_{(2,A)}, \ldots, x_{(n,B)}\right)$. Let

$$X_i^{\alpha} = \frac{x_{(i,\alpha)}}{x_{(i,\alpha)} + x_{(i,\bar{\alpha})}}, \qquad X_i^{\bar{\alpha}} = \frac{x_{(i,\bar{\alpha})}}{x_{(i,\alpha)} + x_{(i,\bar{\alpha})}}.$$

Thus, given that a player has class j, X_j^{α} is the probability that he uses custom k_{α}. We then have

$$\phi_{ii'}(\alpha, \mathbf{x}) = \sum_{i',j'} \Big\{ \left[p(i',j';\bar{\alpha},\alpha|i,j;\alpha,\alpha) + p(i',j';\bar{\alpha},\bar{\alpha}|i,j;\alpha,\alpha) \right] X_j^{\alpha}$$
$$+ \left[p(i',j';\bar{\alpha},\alpha|i,j;\alpha,\bar{\alpha}) + p(i',j';\bar{\alpha},\bar{\alpha}|i,j;\alpha,\bar{\alpha}) \right] X_j^{\bar{\alpha}} \Big\},$$

where the transition probabilities, $p(\cdot|\cdot)$, are taken as conditional on player-I using an aspiration test. The relevant formulae for these conditional probabilities are given in Appendix B, (B1.15, 7), and (B1.5, 6, $\bar{6}$, 7, $\bar{7}$). Using these formulae, together with (B1.2) and (B1.6), we obtain

$$\phi_{ii'}(\alpha, \mathbf{x}) = \frac{1}{4} \sum_{i'} \gamma_{ii'}$$
$$\phi_{ii'}(\alpha, \mathbf{x}) = \left[\gamma_{i2} k_{\alpha}(j,i) + \gamma_{i3} k_{\alpha}(i,j) \right] X_j^{\alpha}$$
$$+ \left[\gamma_{i1} k_{\alpha}(i,j) k_{\alpha}(j,i) + \gamma_{i2} k_{\alpha}(j,i) k_{\alpha}(j,i) + \gamma_{i3} k_{\alpha}(j,i) k_{\alpha}(i,j) + \gamma_{i4} k_{\alpha}(i,j) k_{\alpha}(i,j) \right] X_j^{\bar{\alpha}}$$
$$(i = j).$$

Weshallcomputethenetflowofthesecustomtransitionprobabilities $\Delta\phi_{i,j}(x) = \phi_{ii'}(A, x) - \phi_{ii'}(B, x))$ in the context of our example, that is, when when $(i, j) = (1, 2)$ and $\{A, B\} = \{1, 33\}$. Remember that, when $\{A, B\} = \{1, 33\}$, we have $k_A(1,2) = 0$ and $k_B(1,2) = 1$. Thus,

$$\phi_{12}(1, \mathbf{x}) = \gamma_{12} X_2^1 + \gamma_{13} X_2^{33}; \qquad \phi_{12}(33, \mathbf{x}) = \gamma_{14} X_2^{33} + \gamma_{11} X_2^1;$$
$$\phi_{21}(1, \mathbf{x}) = \gamma_{24} X_1^1 + \gamma_{21} X_1^{33}; \qquad \phi_{21}(33, \mathbf{x}) = \gamma_{22} X_1^{33} + \gamma_{23} X_1^1.$$

from which we obtain:

$$\Delta\phi_{12}(\mathbf{x}) = \frac{1}{6} \eta \left[X_2^{33} - X_2^1 \right],$$
$$\Delta\phi_{21}(\mathbf{x}) = \frac{1}{6} \left[2X_1^{33} - \eta (X_1^1 + X_1^{33}) \right].$$

The above analisys refers to the *out of equibrium* behaviour of the two-custom society, which has been completely neglected in our considerations so far. Note that if $X_2^{II} = X_2^{I}$ then $\Delta\phi_{12}(x) = 0$. On the other hand, if $X_1^{I} = X_1^{II}$ then $\Delta\phi_{21}(x) > 0$, for any $x \in (0,1)$. We have here a way to discriminate between the Immobile Custom and k_{11}. Although k_{11} exhibits the same mobility in equilibrium in the one-custom case, (and prescribes the same behaviour in five out of six cases), out of equilibrium (i.e. when a player of class 2 meets an opponent of class 1), it prescibes a more efficient behaviour for the higher-class player. The latter is in fact the one who is more likely to change her custom, since she has more to loose in the encounter: if she cooperates against a lower-class opponent, she can avoid the inefficient outcome (1,1). This in turn will reduce the probability of changing her custom (measured by $\phi_{21}(k, x)$), producing the slight preference for custom k_{11} exhibited by our simulations.

7. The full 64-custom society

In this final section we consider the case of a society in which any of the 64 possible customs may be present. As in the two-custom case, we suppose that players apply a coordination test or an aspiration test. However, whereas in the two custom case, an agent changing her custom must change to the only other alternative available, in the full system there are many possible choices. We shall assume that custom changes (for whatever reason) are effected only by *local modification*. Thus, if player-I's state prior to the game is (i, k_I), and player-II's is (j, k_{II}), then player-I's strategy in the game is to Defect with probability $k_I(i, j)$. If player-I has cause to change her custom as a result of this experience, then *she only modifies her* (i, j)*-response and nothing else*; *i.e.* the change of custom is $k_I \to k_I'$, where

$$k_I'(i', j') = \begin{cases} k_I(j, i) & \text{if } (i', j') = (i, j) \text{ or } (j, i) \\ k_I(i', j') & \text{otherwise} \end{cases} \qquad (7.1)$$

Notice that $k_I' = k_I$ if $i = j$, so that two players of the same class never modify their customs in response to the game outcome. This is in contrast to the two-custom case discussed previously.

We call such a change a *local* modification because it depends only on the information available to player-I in the particular game, namely the prior class types of the players, (i, j), and the posterior class type of player-I. For example, if there is a failure of coordination at (i, j) (see (6.2)), then, necessarily $i = j$, and player-II will make an unexpected move (Cooperate instead of Defect, or *vice-versa*). If player-I applies a coordination test, then she will attempt to coordinate with player-II in future by coordinating with whatever strategy player-II played when the same (i, j) situation arises again. Similarly, if the custom change is the result of an aspiration test, then player-I reasons that, in order to do better in a similar situation next time, she should play the alternative strategy. This leads to the rule (7.1). In doing this, player-I does not make any assumption about what player-II will do in response to a coordination failure. This is because she only ever knows player-II's class, and not what custom he might be using. Her prior working assumption is always that player-II's custom is the same as hers. If she didn't make this assumption, then the notion of a custom as a coordinating device would lose its force; she might just as well pick a strategy at random.

The result of a game, together with the application of coordination or aspiration tests and local modification with per-player probabilities $(1 - \lambda)$ and λ, respectively, is a transition of the form, $((i, k_I), (j, k_{II})) \to ((i', k'_I), (j', k'_{II}))$. We denote by

$$p(i', j'; k'_I, k'_{II} | i, j; k_I, k_{II}) \tag{7.2}$$

the probability with which such a transition occurs. We compute the transition probabilities (7.2) in Appendix C. Again we have

PROPOSITION 7.1. For $0 < \lambda, \mu < 1$ and $N \geq 64 \times 4 = 256$, the 64-custom Markov process defined on the lattice $\Delta_N \subset \Delta^{255}$, is ergodic.

Proof. See Appendix D. □

For simulation purposes in this section, we use the form (6.3) for γ_{ij}. We are mainly interested in checking whether the push in favour of "more mobile" customs we observed in the two-custom case still operates in the full 64-custom society. Figure 7.1 displays graphically the summary statistics of a set of 400 simulations, 100 for each of the parameter settings $\{\lambda, \kappa\}$, which we used for the two-custom society:

FIGURE 7.1. *The full 64-custom society*

Customs are ordered with respect of their mobility index $B(k)$ evaluated at the estimated equilibrium class distribution of the corresponding 64-custom simulation. As can be spotted from the graphs, the trend toward more mobile customs is evident in all cases. Moreover, this preference increases with both λ and κ as in the two-custom case.

8. Conclusions

Our research program is clearly at a preliminary stage, and the reader may feel uncomfortable finding *ad hoc* assumptions every now and then. Our first (and cheap-talk) justification invokes simplicity and mathematical tractability: the model appears to be complicated enough, even with all these (some would argue) quite implausible short-cuts.

Still, we claim to have some ground for further defence. This is why we tried to justify each assumption on the basis of some plausible intuition (this is, at least, the authors' hope). However, we devote these concluding remarks to point out some critical points, which should be interpreted as guidelines for future research.

Non-equilibrium behaviours. We restrict our attention to the set of behaviours specified in definition 2.2 as ”customs”. This assumptions is, of course, not innocent, and we have no reason to conjecture that, once we allowed a larger set of possible behaviours, our conclusions would not differ in a substantial way. Think, for example, of a sub-population of ’die-harders’, prone to defect regardless of the identity of their opponents: to what extent would their presence affect the dynamics of the system? Or, alternatively, think of sub-populations of ’mixers’, playing a mixed strategy all the time, or ’doves’, cooperating with anybody, etc... All these possibilities are ruled out by Definition 2.2, and we simply do not attempt to predict what would happen if the set of possible behaviours were enlarged substantially.

"Smoother" class ranking In our model, the class of a player is simply the payoff received the last round she has been called to play. In other words, we allow the possibility that a player moves from the top of the social ranking to the bottom (or vice versa) within a single period. This feature of the model is indeed unrealistic, and it could be modified, for example, if we defined the class as some weighted average of the last n payoffs received in the Class Game.

The role of memory. A similar remark can be addressed to the structure of the learning process. Our players have no memory, and every comparison is made with respect to current payoffs: a grain of 'bad luck' could completely upset the *weltanshaung* of a player, regardless any other consideration. Alternative sets of assumptions could design the learning process in a more realistic way; for instance, we might let players apply our coordination and aspiration tests on a longer string of outcomes ("don't let your choices be driven by your last impression!"). Our conjecture is that a modification in this direction should not change our conclusions in a substantial way, but we are not able, at this stage, to provide a formal justification of this claim.

More complex Class Games. One could argue that some of our results crucially depend upon the particular features of the Class Game we have chosen, namely: *chicken.* It would be interesting to apply the same analysis to more complex strategic frameworks. For example, a natural extension of the model would be to the classic *Nash Demand Game*, where the strategic framework of *chicken* is extended to a much richer strategy space for each player.

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Appendix A - One-custom society.

A1. Proof ofProposition 5.1. From equations (5.1), we obtain the explicitform for the discrete dynamic (5.4)

$$x_1' = x_3' = \frac{1}{4}||\mathbf{x}||^2, \tag{A1.1a}$$

$$x_2' = x_4' = \frac{1}{2} - \frac{1}{4}||\mathbf{x}||^2, \tag{A1.1b}$$

where we have written $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{x}' = \mathbf{x}(t+1)$, and $||\mathbf{x}|| = \sum_i x_i^2$. Note that this dynamic is independent of the custom k, and so any equilibria will be universal.

Clearly, after at most one time step, the dynamic is confined to the invariant subspace, $x_1 = x_3$, $x_2 = x_4$. We also have the constraint, $\sum_i x_i = 1$. Thus, in this constarined space, $||\mathbf{x}||^2 = 2\left(x_1^2 + \left(\frac{1}{2} - x_1\right)^2\right) = \frac{1}{2} - 2x_1 + 4x_1^2$. The resulting constrained dynamic is therefore 1-dimensional, and can be written in the form,

$$x' - x = \frac{1}{8} - \frac{3}{2}x + x^2, \tag{A1.2}$$

where $x = x_1$. The quadratic factorizes as, $(x - x_+)(x - x_-)$, where $x_\pm = \frac{1}{4}(3 \pm \sqrt{7})$. The first factor is always negative since $x_+ > 1$, and the second factor is negative if $0 \leq x < x_-$, and positive if $x_- < x \leq 1$. It follows that x_- is the unique global attractor for this dynamic, and hence that $\bar{\mathbf{x}} = \left(\frac{1}{4}(3 - \sqrt{7}), \frac{1}{4}(\sqrt{7} - 1), \frac{1}{4}(3 - \sqrt{7}), \frac{1}{4}(\sqrt{7} - 1)\right)$ is the unique global attractor for the dynamic (A1.1). □

A2. Proof of Proposition 5.2. Using the formula (5.1) for the transition probabilities, we can compute the Bartholemew mobility index for each custom, and order the customs by increasing mobility. The resulting order is shown in Table A.1.

Note that the custom k_1 (together with custom k_{33} has mobility index which is strictly less than any other custom, and the Liberal custom has an index which is strictly greater than any other. Table A.1 therefore constitutes a proof of Proposition 5.2. □

#	$B(k)$	#	$B(k)$	#	$B(k)$	#	$B(k)$
1	0.677124	22	0.838562	3	1.35425	24	1.51569
33	0.677124	25	0.838562	35	1.35425	27	1.51569
17	0.692811	54	0.838562	19	1.36994	56	1.51569
49	0.692811	57	0.838562	51	1.36994	59	1.51569
2	0.75	10	0.895751	4	1.42712	12	1.57288
5	0.75	13	0.895751	7	1.42712	15	1.57288
34	0.75	42	0.895751	36	1.42712	44	1.57288
37	0.75	45	0.895751	39	1.42712	47	1.57288
18	0.765687	26	0.911438	20	1.44281	28	1.58856
21	0.765687	29	0.911438	23	1.44281	31	1.58856
50	0.765687	58	0.911438	52	1.44281	60	1.58856
53	0.765687	61	0.911438	55	1.44281	63	1.58856
6	0.822876	14	0.968627	8	1.5	16	1.64575
9	0.822876	46	0.968627	11	1.5	48	1.64575
38	0.822876	30	0.984313	40	1.5	32	1.66144
41	0.822876	62	0.984313	43	1.5	64	1.66144

TABLE A.1. *Customs ordered by increasing Bartholomew mobility index.*

Appendix B: Two-Custom Society

B1. Explicit forms for transition probabilities and deterministic equations.

To compute the transition probabilities (6.1), we must consider four cases.

Case 1. Both players apply a coordination test. This occurs with probability $(1-\lambda)^2$. In this case the possible *non-zero* transition probabilities are:

$$p(i',j';\alpha,\beta|i,j;\alpha,\beta) = \delta_{\alpha,\beta}(i,j)p_\alpha(i',j'|i,j) \qquad (B1.1a)$$

$$p(i',j';\bar\alpha,\bar\beta|i,j;\alpha,\beta) = \bar\delta_{\alpha,\beta}(i,j)\bar p_\alpha(i',j'|i,j) \qquad (B1.1b)$$

Here, $\delta_{\alpha,\beta}$ is the *coordination index* function: $\delta_{\alpha,\beta}(i,j) = 1$ if α and β coordinate at (i,j), and $\delta_{\alpha,\beta}(i,j) = 0$ otherwise. Also, $\bar\delta_{\alpha,\beta}(i,j) = 1 - \delta_{\alpha,\beta}(i,j)$, so that α and β fail to coordinate at (i,j) if and only if $\bar\delta_{\alpha,\beta}(i,j) = 1$. Clearly, $\delta_{\alpha,\beta}(i,j) = \delta_{\alpha,\beta}(j,i)$. The probabilities $p_\alpha(i',j'|i,j)$ and $\bar p_\alpha(i',j'|i,j)$, are the coordinated and uncoordinated transition prbabilities, respectively; *i.e.*

$$p_\alpha(i',j'|i,i) = \frac{1}{4}\left(\delta^1_{i'}\delta^1_{j'} + \delta^2_{i'}\delta^4_{j'} + \delta^3_{i'}\delta^3_{j'} + \delta^4_{i'}\delta^2_{j'}\right) \qquad (B1.2a)$$

$$p_\alpha(i',j'|i,j) = h_\alpha(j,i)\delta^2_{i'}\delta^4_{j'} + h_\alpha(i,j)\delta^2_{j'}\delta^4_{i'} \qquad (i = j) \qquad (B1.2b)$$

$$\bar p_\alpha(i',j'|i,i) = 0 \qquad (B1.2c)$$

$$\bar p_\alpha(i',j'|i,j) = h_\alpha(i,j)\delta^1_{i'}\delta^1_{j'} + h_\alpha(j,i)\delta^3_{i'}\delta^3_{j'} \qquad (i = j) \qquad (B1.2d)$$

Case 2. Player-I uses a coordination test and player-II uses an aspiration test. This occurs with probability $(1-\lambda)\lambda$. The possible non-zero transition probabilities are

$$p(i',j';\alpha,\beta|i,j;\alpha,\beta) = \delta_{\alpha,\beta}(i,j)(1-\gamma_{jj'})p_\alpha(i',j'|i,j) \qquad (B1.3a)$$

$$p(i',j';\alpha,\bar\beta|i,j;\alpha,\beta) = \delta_{\alpha,\beta}(i,j)\gamma_{jj'}\cdot p_\alpha(i',j'|i,j) \qquad (B1.3b)$$

$$p(i',j';\bar\alpha,\beta|i,j;\alpha,\beta) = \bar\delta_{\alpha,\beta}(i,j)(1-\gamma_{jj'})\bar p_\alpha(i',j'|i,j) \qquad (B1.3c)$$

$$p(i',j';\bar\alpha,\bar\beta|i,j;\alpha,\beta) = \bar\delta_{\alpha,\beta}(i,j)\gamma_{jj'}\cdot \bar p_\alpha(i',j'|i,j) \qquad (B1.3d)$$

Case 3. Player-I uses an aspiration test and player-II uses a coordination test. This occurs with probability $\lambda(1-\lambda)$. The possible non-zero transition probabilities are

$$p(i',j';\alpha,\beta|i,j;\alpha,\beta) = \delta_{\alpha,\beta}(i,j)(1-\gamma_{ii'})p_\alpha(i',j'|i,j) \qquad (B1.4a)$$

$$p(i', j'; \bar{\alpha}, \beta \mid i, j; \alpha, \beta) = \delta_{\alpha,\beta}(i, j)\gamma_{ii'}\, p_{\alpha}(i', j' \mid i, j) \qquad (B1.4b)$$

$$p(i', j'; \alpha, \bar{\beta} \mid i, j; \alpha, \beta) = \bar{\delta}_{\alpha,\beta}(i, j)(1 - \gamma_{ii'})\bar{p}_{\alpha}(i', j' \mid i, j) \qquad (B1.4c)$$

$$p(i', j'; \bar{\alpha}, \bar{\beta} \mid i, j; \alpha, \beta) = \bar{\delta}_{\alpha,\beta}(i, j)\gamma_{ii'}\, \bar{p}_{\alpha}(i', j' \mid i, j) \qquad (B1.4d)$$

Case 4. Both players use an aspiration test. This occurs with probability λ^2. The possible non-zero transition probabilities are

$$p(i', j'; \alpha, \beta \mid i, j; \alpha, \beta) = \delta_{\alpha,\beta}(i, j)(1 - \gamma_{ii'})(1 - \gamma_{jj'})p_{\alpha}(i', j' \mid i, j) \qquad (B1.5a)$$

$$p(i', j'; \alpha, \beta \mid i, j; \alpha, \beta) = \bar{\delta}_{\alpha,\beta}(i, j)(1 - \gamma_{ii'})(1 - \gamma_{jj'})\bar{p}_{\alpha}(i', j' \mid i, j) \qquad (B1.5\bar{a})$$

$$p(i', j'; \bar{\alpha}, \beta \mid i, j; \alpha, \beta) = \delta_{\alpha,\beta}(i, j)\gamma_{ii'}(1 - \gamma_{jj'})p_{\alpha}(i', j' \mid i, j) \qquad (B1.5b)$$

$$p(i', j'; \bar{\alpha}, \beta \mid i, j; \alpha, \beta) = \bar{\delta}_{\alpha,\beta}(i, j)\gamma_{ii'}(1 - \gamma_{jj'})\bar{p}_{\alpha}(i', j' \mid i, j) \qquad (B1.5\bar{b})$$

$$p(i', j'; \alpha, \bar{\beta} \mid i, j; \alpha, \beta) = \delta_{\alpha,\beta}(i, j)(1 - \gamma_{ii'})\gamma_{jj'}\, p_{\alpha}(i', j' \mid i, j) \qquad (B1.5c)$$

$$p(i', j'; \alpha, \bar{\beta} \mid i, j; \alpha, \beta) = \bar{\delta}_{\alpha,\beta}(i, j)(1 - \gamma_{ii'})\gamma_{jj'}\, \bar{p}_{\alpha}(i', j' \mid i, j) \qquad (B1.5\bar{c})$$

$$p(i', j'; \bar{\alpha}, \bar{\beta} \mid i, j; \alpha, \beta) = \delta_{\alpha,\beta}(i, j)\gamma_{ii'}\gamma_{jj'}\, p_{\alpha}(i', j' \mid i, j) \qquad (B1.5d)$$

$$p(i', j'; \bar{\alpha}, \bar{\beta} \mid i, j; \alpha, \beta) = \bar{\delta}_{\alpha,\beta}(i, j)\gamma_{ii'}\gamma_{jj'}\, \bar{p}_{\alpha}(i', j' \mid i, j) \qquad (B1.5\bar{d})$$

We can now compute the *unconditional* transition probabilities (6.1). For example,

$$p(i', j'; \alpha, \beta \mid i, j; \alpha, \beta) = (1-\lambda)^2(B1.1a) + (1-\lambda)\lambda(B1.3a) + \lambda(1-\lambda)(B1.4a) + \lambda^2(B1.5a).$$

We obtain thus,

$$p(i', j'; \alpha, \beta \mid i, j; \alpha, \beta) =$$

$$(1 - \lambda\gamma_{ii'})(1 - \lambda\gamma_{jj'})\delta_{\alpha,\beta}(i, j)p_{\alpha}(i', j' \mid i, j) + \lambda^2(1 - \gamma_{ii'})(1 - \gamma_{jj'})\bar{\delta}_{\alpha,\beta}(i, j)\bar{p}_{\alpha}(i', j' \mid i, j) \qquad (B1.6a)$$

$$p(i', j'; \alpha, \bar{\beta} \mid i, j; \alpha, \beta) =$$

$$(1 - \lambda\gamma_{ii'})\lambda\gamma_{jj'}\delta_{\alpha,\beta}(i, j)p_{\alpha}(i', j' \mid i, j) + \lambda(1 - \gamma_{ii'})(1 - \lambda(1 - \gamma_{jj'}))\bar{\delta}_{\alpha,\beta}(i, j)\bar{p}_{\alpha}(i', j' \mid i, j) \qquad (B1.6b)$$

$$p(i', j'; \bar{\alpha}, \beta \mid i, j; \alpha, \beta) =$$

$$\lambda\gamma_{ii'}(1-\lambda\gamma_{jj'})\delta_{\alpha,\beta}(i,j)p_\alpha(i',j'|i,j) + (1-\lambda(1-\gamma_{ii'}))\lambda(1-\gamma_{jj'})\bar{\delta}_{\alpha,\beta}(i,j)\bar{p}_\alpha(i',j'|i,j)$$

(B1.6c)

$$p(i',j';\bar{\alpha},\bar{\beta}|i,j;\alpha,\beta) =$$

$$\lambda^2\gamma_{ii'}\gamma_{jj'}\delta_{\alpha,\beta}(i,j)p_\alpha(i',j'|i,j) + (1-\lambda(1-\gamma_{ii'}))(1-\lambda(1-\gamma_{jj'}))\bar{\delta}_{\alpha,\beta}(i,j)\bar{p}_\alpha(i',j'|i,j)$$

(B1.6d)

We now compute an explicit form for the deterministic equations (6.4). Note from equations (B1.2) that

$$\sum_{j'} p_\alpha(i',j'|i,i) = \frac{1}{4} \qquad (B1.7a)$$

$$\sum_{j'} p_\alpha(i',j'|i,j) = h_\alpha(j,i)\delta^2_{i'} + h_\alpha(i,j)\delta^1_{i'} \qquad (i = j) \qquad (B1.7b)$$

$$\sum_{j'} \bar{p}_\alpha(i',j'|i,i) = 0 \qquad (B1.7c)$$

$$\sum_{j'} \bar{p}_\alpha(i',j'|i,j) = h_\alpha(i,j)\delta^1_{i'} + h_\alpha(j,i)\delta^2_{i'} \qquad (i = j) \qquad (B1.7d)$$

Also, noting that $h_\alpha(i,j)^2 = h_\alpha(i,j)$ and $h_\alpha(i,j)h_\alpha(j,i) = 0$, for $i = j$, it is easy to check that, for $i = j$,

$$\delta_{\alpha,\beta}(i,j)h_\alpha(i,j) = \delta_{\alpha,\beta}(i,j)h_\beta(i,j) = h_\alpha(i,j)h_\beta(i,j) \qquad (B1.8a)$$

$$\bar{\delta}_{\alpha,\beta}(i,j)h_\alpha(i,j) = \bar{\delta}_{\alpha,\beta}(i,j)h_\beta(j,i) = h_\alpha(i,j)h_\beta(j,i) \qquad (B1.8b)$$

It therefore follows from equations (B1.7) and (B1.8) that

$$\sum_{j'} \delta_{\alpha,\beta}(i,i)p_\alpha(i',j'|i,i) = \frac{1}{4} \qquad (B1.9a)$$

$$\sum_{j'} \delta_{\alpha,\beta}(i,j)p_\alpha(i',j'|i,j) = h_\alpha(j,i)h_\beta(j,i)\delta^2_{i'} + h_\alpha(i,j)h_\beta(i,j)\delta^1_{i'} \qquad (i = j) \qquad (B1.9b)$$

$$\sum_{j'} \bar{\delta}_{\alpha,\beta}(i,i)\bar{p}_\alpha(i',j'|i,i) = 0 \qquad (B1.9c)$$

$$\sum_{j'} \bar{\delta}_{\alpha,\beta}(i,j)\bar{p}_\alpha(i',j'|i,j) = h_\alpha(i,j)h_\beta(j,i)\delta^1_{i'} + h_\alpha(j,i)h_\beta(i,j)\delta^2_{i'} \qquad (i = j) \qquad (B1.9d)$$

Using equations (B1.6) and (B1.9), we first obtain an explicit form for equations (6.4) with $\rho = A$. Thus, equations (6.4) can be written

$$
\begin{aligned}
\frac{dx_{(i,A)}}{dt} = & -2x_{(i,A)} \\
& + 2\sum_{i,j}\left\{\sum_{j'}\left[p(i',j';A,A|i,j;A,A) + p(i',j';A,B|i,j;A,A)\right]\right\}x_{(i,A)}x_{(j,A)} \\
& + 2\sum_{i,j}\left\{\sum_{j'}\left[p(i',j';A,A|i,j;A,B) + p(i',j';A,B|i,j;A,B)\right]\right\}x_{(i,A)}x_{(j,B)} \\
& + 2\sum_{i,j}\left\{\sum_{j'}\left[p(i',j';A,A|i,j;B,A) + p(i',j';A,B|i,j;B,A)\right]\right\}x_{(i,B)}x_{(j,A)} \\
& + 2\sum_{i,j}\left\{\sum_{j'}\left[p(i',j';A,A|i,j;B,B) + p(i',j';A,B|i,j;B,B)\right]\right\}x_{(i,B)}x_{(j,B)}
\end{aligned}
\qquad (B1.10)
$$

Now, from (B1.6a,b) and (B1.3c) we have

$$
\begin{aligned}
& p(i',j';A,A|i,i;A,A) + p(i',j';A,B|i,i;A,A) \\
& \qquad = \left[(1-\lambda\gamma_{ii})(1-\lambda\gamma_{ii}) + (1-\lambda\gamma_{ii})\lambda\gamma_{ii}\right]\delta_{A,A}(i,j)p_A(i',j'|i,i) \\
& \qquad = (1-\lambda\gamma_{ii})\delta_{A,A}(i,j)p_A(i',j'|i,i)
\end{aligned}
$$

Thus, from (B1.9a) we obtain

$$
\sum_{j'}\left[p(i',j';A,A|i,i;A,A) + p(i',j';A,B|i,i;A,A)\right] = \frac{1}{4}(1-\lambda\gamma_{ii}) \qquad (B1.11a)
$$

Similar calculations yield

$$
\sum_{j'}\left[p(i',j';A,A|i,i;A,B) + p(i',j';A,B|i,i;A,B)\right] = \frac{1}{4}(1-\lambda\gamma_{ii}) \qquad (B1.11b)
$$

$$
\sum_{j'}\left[p(i',j';A,A|i,i;B,A) + p(i',j';A,B|i,i;B,A)\right] = \frac{1}{4}\lambda\gamma_{ii} \qquad (B1.11c)
$$

$$
\sum_{j'}\left[p(i',j';A,A|i,i;B,B) + p(i',j';A,B|i,i;B,B)\right] = \frac{1}{4}\lambda\gamma_{ii} \qquad (B1.11d)
$$

From equations (B1.11) we can now pull out the term in the summations on the right hand side of (B1.10) for which $i = j$, to obtain

$$
\frac{1}{2}||\mathbf{x}_A||^2 + \frac{1}{2}\mathbf{x}_A\cdot\mathbf{x}_B - \lambda\frac{1}{2}\sum_i \gamma_{ii}\left[x^2_{(i,A)} - x^2_{(i,B)}\right] \qquad (B1.12)
$$

where, as usual, $\mathbf{u}\cdot\mathbf{v} = \sum_i u_i v_i$, and $||\mathbf{u}||^2 = \mathbf{u}\cdot\mathbf{u}$.

We now turn to the off-diagonal cases, $i = j$. From (B1.6a,b), we have

$$p(i',j';A,A|i,j;A,A) + p(i',j';A,B|i,j;A,A)$$

$$= (1-\lambda\gamma_{ij})\delta_{A,A}(i,j)p_A(i',j'|i,j) + \lambda(1-\gamma_{ij})\bar{\delta}_{A,A}(i,j)\bar{p}_A(i',j'|i,j)$$

and using (B1.7b,d) gives

$$\sum_{j'} \left[p(i',j';A,A|i,j;A,A) + p(i',j';A,B|i,j;A,A)\right] = (1-\lambda\gamma_{ij})\left[k_A(j,i)\delta_j^2 + k_A(i,j)\delta_j^4\right] \tag{B1.13a}$$

[Remember that $k_\alpha(i,j)^2 = k_\alpha(i,j)$ and $k_\alpha(i,j)k_\alpha(j,i) = 0$.] Similar calculations yield

$$\sum_{j'} \left[p(i',j';A,A|i,j;A,B) + p(i',j';A,B|i,j;A,B)\right] =$$

$$(1-\lambda\gamma_{ij})\left[k_A(j,i)k_B(j,i)\delta_j^2 + k_A(i,j)k_B(i,j)\delta_j^4\right]$$

$$+ \lambda(1-\gamma_{ij})\left[k_A(i,j)k_B(j,i)\delta_j^1 + k_A(j,i)k_B(i,j)\delta_j^3\right] \tag{B1.13b}$$

$$\sum_{j'} \left[p(i',j';A,A|i,j;B,A) + p(i',j';A,B|i,j;B,A)\right] =$$

$$\lambda\gamma_{ij}\left[k_B(j,i)k_A(j,i)\delta_j^2 + k_B(i,j)k_A(i,j)\delta_j^4\right]$$

$$+ \left(1-\lambda(1-\gamma_{ij})\right)\left[k_B(i,j)k_A(j,i)\delta_j^1 + k_B(j,i)k_A(i,j)\delta_j^3\right] \tag{B1.13c}$$

$$\sum_{j'} \left[p(i',j';A,A|i,j;B,B) + p(i',j';A,B|i,j;B,B)\right] = \lambda\gamma_{ij}\left[k_B(j,i)\delta_j^2 + k_B(i,j)\delta_j^4\right] \tag{B1.13d}$$

Now multiply (B1.13b) by $\pi_{(i,A)}\pi_{(j,B)}$ and sum over $i = j$, then multiply (B1.13c) by $\pi_{(i,B)}\pi_{(j,A)}$, sum over $i = j$ and interchange the dummy indices, $i \leftrightarrow j$, to obtain

$$\sum_{i=j}\left\{\sum_{j'} \left[p(i',j';A,A|i,j;A,B) + p(i',j';A,B|i,j;A,B)\right]\right\}\pi_{(i,A)}\pi_{(j,B)}$$

$$+ \sum_{i=j}\left\{\sum_{j'} \left[p(i',j';A,A|i,j;B,A) + p(i',j';A,B|i,j;B,A)\right]\right\}\pi_{(i,B)}\pi_{(j,A)}$$

$$=$$

$$\sum_{i=j}\left\{k_A(i,j)k_B(j,i)\delta_j^1 + k_A(j,i)k_B(j,i)\delta_j^2 + k_A(j,i)k_B(i,j)\delta_j^3 + k_A(i,j)k_B(i,j)\delta_j^4\right\}\pi_{(i,A)}\pi_{(j,B)}$$

$$-\lambda\sum_{i=j}\left\{(\gamma_{ij} - \gamma_{ji})\left[k_A(i,j)k_B(j,i)\delta_j^1 + k_A(j,i)k_B(i,j)\delta_j^3\right] + \right.$$

$$\left.\left[\gamma_{ij}k_A(j,i)k_B(j,i) - \gamma_{ji}k_A(i,j)k_B(i,j)\right]\delta_j^2 + \left[\gamma_{ij}k_A(i,j)k_B(i,j) - \gamma_{ji}k_A(j,i)k_B(j,i)\right]\delta_j^4\right\}\pi_{(i,A)}\pi_{(j,B)} \tag{B1.14}$$

Finally, multiplying (B1.13a) by $w_{(i,A)}w_{(j,A)}$, (B1.13d) by $w_{(i,B)}w_{(j,B)}$, and summing over $i = j$, we obtain terms

$$\sum_{i=j}\left\{\sum_{\downarrow}\left|\rho(i',j';A,A|i,j;A,A) + \rho(i',j';A,B|i,j;A,A)\right|\right\}w_{(i,A)}w_{(j,A)}$$

$$= \sum_{i=j}(1-\lambda\gamma_{i1})\left|k_A(j,i)\delta_1^2 + k_A(i,j)\delta_1^{*}\right| w_{(i,A)}w_{(j,A)} \qquad (B1.15)$$

$$\sum_{i=j}\left\{\sum_{\downarrow}\left|\rho(i',j';A,A|i,j;B,B) + \rho(i',j';A,B|i,j;B,B)\right|\right\}w_{(i,B)}w_{(j,B)}$$

$$= \lambda\sum_{i=j}\gamma_{i1}\left|k_B(j,i)\delta_1^2 + k_B(i,j)\delta_1^{*}\right| w_{(i,B)}w_{(j,B)} \qquad (B1.16)$$

By combining (B1.12), (B1.14), (B1.15) and (B1.16), we can now write down explicit forms for equations (B1.10), to obtain:

$$\frac{dw_{(1,A)}}{dt} = -2w_{(1,A)} + \frac{1}{2}||\mathbf{x}_A||^2 + \frac{1}{2}\mathbf{x}_A\cdot\mathbf{x}_B + 2\sum_{i=j}k_A(i,j)k_B(j,i)w_{(i,A)}w_{(j,B)}$$
$$-\lambda\left\{\frac{1}{2}\sum_{i}\gamma_{i1}\left|w^2_{(i,A)} - w^2_{(i,B)}\right| + 2\sum_{i=j}\left|\gamma_{i1}-\gamma_{j1}\right|k_A(i,j)k_B(j,i)w_{(i,A)}w_{(j,B)}\right\}$$
$$(B1.17a)$$

$$\frac{dw_{(2,A)}}{dt} = -2w_{(2,A)} + \frac{1}{2}||\mathbf{x}_A||^2 + \frac{1}{2}\mathbf{x}_A\cdot\mathbf{x}_B$$
$$+ 2\sum_{i=j}k_A(j,i)w_{(i,A)}w_{(j,A)} + 2\sum_{i=j}k_A(j,i)k_B(j,i)w_{(i,A)}w_{(j,B)}$$
$$-\lambda\left\{\frac{1}{2}\sum_{i}\gamma_{i2}\left|w^2_{(i,A)} - w^2_{(i,B)}\right| + 2\sum_{i=j}\gamma_{i2}\left|k_A(j,i)w_{(i,A)}w_{(j,A)} - k_B(j,i)w_{(i,B)}w_{(j,B)}\right|\right.$$
$$\left. + 2\sum_{i=j}\left|\gamma_{i2}k_A(j,i)k_B(j,i) - \gamma_{j2}k_A(i,j)k_B(i,j)\right| w_{(i,A)}w_{(j,B)}\right\}$$
$$(B1.17b)$$

$$\frac{dw_{(3,A)}}{dt} = -2w_{(3,A)} + \frac{1}{2}||\mathbf{x}_A||^2 + \frac{1}{2}\mathbf{x}_A\cdot\mathbf{x}_B + 2\sum_{i=j}k_A(j,i)k_B(i,j)w_{(i,A)}w_{(j,B)}$$
$$-\lambda\left\{\frac{1}{2}\sum_{i}\gamma_{i3}\left|w^2_{(i,A)} - w^2_{(i,B)}\right| + 2\sum_{i=j}\left|\gamma_{i3}-\gamma_{j3}\right|k_A(j,i)k_B(i,j)w_{(i,A)}w_{(j,B)}\right\}$$
$$(B1.17c)$$

$$\frac{dw_{(4,A)}}{dt} = -2w_{(4,A)} + \frac{1}{2}||\mathbf{x}_A||^2 + \frac{1}{2}\mathbf{x}_A\cdot\mathbf{x}_B$$
$$+ 2\sum_{i=j}k_A(i,j)w_{(i,A)}w_{(j,A)} + 2\sum_{i=j}k_A(i,j)k_B(i,j)w_{(i,A)}w_{(j,B)}$$

$$- \lambda \left\{ \frac{1}{2} \sum_{i} \gamma_{i4} \left| x^2_{(i,A)} - x^2_{(i,B)} \right| + 2 \sum_{i \neq j} \gamma_{i4} \left| k_A(i,j) x_{(i,A)} x_{(j,A)} - k_B(i,j) x_{(i,B)} x_{(j,B)} \right| + 2 \sum_{i \neq j} \left| \gamma_{i4} k_A(i,j) k_B(i,j) - \gamma_{j4} k_A(j,i) k_B(j,i) \right| x_{(i,A)} x_{(j,B)} \right\} \tag{B1.17d}$$

Similarly, the equations for $\mathbf{x}_B$ are:

$$\frac{dx_{(1,B)}}{dt} = -2x_{(1,B)} + \frac{1}{2}\|\mathbf{x}_B\|^2 + \frac{1}{2}\mathbf{x}_A \cdot \mathbf{x}_B + 2 \sum_{i \neq j} k_A(i,j) k_B(j,i) x_{(i,A)} x_{(j,B)} + \lambda \left\{ \frac{1}{2} \sum_{i} \gamma_{i1} \left| x^2_{(i,A)} - x^2_{(i,B)} \right| + 2 \sum_{i \neq j} \left| \gamma_{i1} - \gamma_{j1} \right| k_A(i,j) k_B(j,i) x_{(i,A)} x_{(j,B)} \right\} \tag{B1.18a}$$

$$\frac{dx_{(2,B)}}{dt} = -2x_{(2,B)} + \frac{1}{2}\|\mathbf{x}_B\|^2 + \frac{1}{2}\mathbf{x}_A \cdot \mathbf{x}_B + 2 \sum_{i \neq j} k_B(j,i) x_{(i,B)} x_{(j,B)} + 2 \sum_{i \neq j} k_A(i,j) k_B(i,j) x_{(i,A)} x_{(j,B)} + \lambda \left\{ \frac{1}{2} \sum_{i} \gamma_{i2} \left| x^2_{(i,A)} - x^2_{(i,B)} \right| + 2 \sum_{i \neq j} \gamma_{i2} \left| k_A(j,i) x_{(i,A)} x_{(j,A)} - k_B(j,i) x_{(i,B)} x_{(j,B)} \right| + 2 \sum_{i \neq j} \left| \gamma_{i2} k_A(j,i) k_B(j,i) - \gamma_{j2} k_A(i,j) k_B(i,j) \right| x_{(i,A)} x_{(j,B)} \right\} \tag{B1.18b}$$

$$\frac{dx_{(3,B)}}{dt} = -2x_{(3,B)} + \frac{1}{2}\|\mathbf{x}_B\|^2 + \frac{1}{2}\mathbf{x}_A \cdot \mathbf{x}_B + 2 \sum_{i \neq j} k_A(j,i) k_B(i,j) x_{(i,A)} x_{(j,B)} + \lambda \left\{ \frac{1}{2} \sum_{i} \gamma_{i3} \left| x^2_{(i,A)} - x^2_{(i,B)} \right| + 2 \sum_{i \neq j} \left| \gamma_{i3} - \gamma_{j3} \right| k_A(j,i) k_B(i,j) x_{(i,A)} x_{(j,B)} \right\} \tag{B1.18c}$$

$$\frac{dx_{(4,B)}}{dt} = -2x_{(4,B)} + \frac{1}{2}\|\mathbf{x}_B\|^2 + \frac{1}{2}\mathbf{x}_A \cdot \mathbf{x}_B + 2 \sum_{i \neq j} k_B(i,j) x_{(i,B)} x_{(j,B)} + 2 \sum_{i \neq j} k_A(j,i) k_B(j,i) x_{(i,A)} x_{(j,B)} + \lambda \left\{ \frac{1}{2} \sum_{i} \gamma_{i4} \left| x^2_{(i,A)} - x^2_{(i,B)} \right| + 2 \sum_{i \neq j} \gamma_{i4} \left| k_A(i,j) x_{(i,A)} x_{(j,A)} - k_B(i,j) x_{(i,B)} x_{(j,B)} \right| + 2 \sum_{i \neq j} \left| \gamma_{i4} k_A(i,j) k_B(i,j) - \gamma_{j4} k_A(j,i) k_B(j,i) \right| x_{(i,A)} x_{(j,B)} \right\} \tag{B1.18d}$$

Appendix C: The 64-Custom Society

C1. Transition probabilities for the 64-custom case We compute the transition probabilities (7.2). First note that, because the local modification rule (7.1) can have no effect on customs when $i = j$, we have

$$p(i',j';k'_I,k'_{II}|i,i;k_I,k_{II}) = \begin{cases} p(i',j'|i,i) & \text{when } (k'_I,k'_{II}) = (k_I,k_{II}) \\ 0 & \text{otherwise} \end{cases} \qquad (C1.1)$$

where $p(i',j'|i,i)$ is the elementary (one custom) transition probability given by (3.1a).

Now suppose $i = j$. Let $\alpha = k_I(i,j)$ and $\beta = k_{II}(j,i)$ be the probabilities with which player-I and player-II will Defect. The $\alpha,\beta \in \{0,1\}$. By the local modification rule (7.1), k'_I and k'_{II} depend only on α and β, and not on the values of k_I or k_{II} at any other class-pair. Also, k'_I or k'_{II} can differ from k_I or k_{II} only at (i,j) and (j,i). Denote by ${}^{(i,j)}k$ the custom obtained from k by *local modification at* (i,j); *i.e.*

$${}^{(i,j)}k(i',j') = \begin{cases} k(j,i) & \text{if } (i',j') = (i,j) \text{ or } (j,i) \\ k(i',j') & \text{otherwise} \end{cases} \qquad (C1.2)$$

Clearly ${}^{(i,j)}k = {}^{(j,i)}k$. Then $k'_I = k_I$ or ${}^{(i,j)}k_I$ and $k'_{II} = k_{II}$ or ${}^{(j,i)}k'_{II}$. It follows that a possible transition $((i,k_I),(j,k_{II})) \rightarrow ((i',k'_I),(j',k'_{II}))$ is completely specified by the *local* transition, $((i,\alpha),(j,\beta)) \rightarrow ((i',\alpha'),(j',\beta'))$, where $\alpha' = k'_I(i,j)$ and $\beta' = k'_{II}(j,i)$. In fact, if

$$p(i',j';\alpha',\beta'|i,j;\alpha,\beta) \qquad (C1.3)$$

is the probability of this latter transition, then the possible non-zero probabilities (20) with $i = j$ are given by

$$p(i',j';k_I,k_{II}|i,j;k_I,k_{II}) = p(i',j';\alpha,\beta|i,j;\alpha,\beta) \qquad (C1.4a)$$

$$p(i',j';{}^{(i,j)}k_I,k_{II}|i,j;k_I,k_{II}) = p(i',j';1-\alpha,\beta|i,j;\alpha,\beta) \qquad (C1.4b)$$

$$p(i',j';k_I,{}^{(j,i)}k_{II}|i,j;k_I,k_{II}) = p(i',j';\alpha,1-\beta|i,j;\alpha,\beta) \qquad (C1.4c)$$

$$p(i',j';{}^{(i,j)}k_I,{}^{(j,i)}k_{II}|i,j;k_I,k_{II}) = p(i',j';1-\alpha,1-\beta|i,j;\alpha,\beta) \qquad (C1.4d)$$

It therefore remains to compute the probabilities (C1.3). There are four cases.

Case 1. Both players apply a coordination test. This occurs with probability $(1-\lambda)^2$. Note that a coordination *failure* can occur only if $\alpha = \beta$, and in this case $\alpha' = \beta' = 1-\alpha$. Otherwise, $\alpha' = \alpha$ and $\beta' = \beta = 1-\alpha$. For $\alpha \in \{0,1\}$, define elementary transition probabilities

$$p_\alpha(i',j'|i,j) = (1-\alpha)\delta^2_{i'}\delta^1_{j'} + \alpha\delta^1_{i'}\delta^2_{j'} \qquad (C1.5a)$$

$$\bar{p}_\alpha(i',j'|i,j) = \alpha\delta^1_{i'}\delta^1_{j'} + (1-\alpha)\delta^2_{i'}\delta^2_{j'} \qquad (C1.5b)$$

We then have

$$p(i',j';1-\alpha,1-\alpha|i,j;\alpha,\alpha) = \bar{p}_\alpha(i',j'|i,j) \qquad (C1.6a)$$

$$p(i',j';\alpha,1-\alpha|i,j;\alpha,1-\alpha) = p_\alpha(i',j'|i,j) \qquad (C1.6b)$$

Case 2. Player-I applies a coordination test and player-II applies an aspiration test. This occurs with probability $(1-\lambda)\lambda$. The possible non-zero transition probabilities are

$$p(i',j';1-\alpha,\alpha|i,j;\alpha,\alpha) = (1-\gamma_{jj'})\bar{p}_\alpha(i',j'|i,j) \qquad (C1.7a)$$

$$p(i',j';1-\alpha,1-\alpha|i,j;\alpha,\alpha) = \gamma_{jj'}\bar{p}_\alpha(i',j'|i,j) \qquad (C1.7b)$$

$$p(i',j';\alpha,1-\alpha|i,j;\alpha,1-\alpha) = (1-\gamma_{jj'})p_\alpha(i',j'|i,j) \qquad (C1.7c)$$

$$p(i',j';\alpha,\alpha|i,j;\alpha,1-\alpha) = \gamma_{jj'}p_\alpha(i',j'|i,j) \qquad (C1.7d)$$

Case 3. Player-I applies an aspiration test and player-II applies a coordination test. This occurs with probability $\lambda(1-\lambda)$. The possible non-zero transition probabilities are

$$p(i',j';\alpha,1-\alpha|i,j;\alpha,\alpha) = (1-\gamma_{ii'})\bar{p}_\alpha(i',j'|i,j) \qquad (C1.8a)$$

$$p(i',j';1-\alpha,1-\alpha|i,j;\alpha,\alpha) = \gamma_{ii'}\bar{p}_\alpha(i',j'|i,j) \qquad (C1.8b)$$

$$p(i',j';\alpha,1-\alpha|i,j;\alpha,1-\alpha) = (1-\gamma_{ii'})p_\alpha(i',j'|i,j) \qquad (C1.8c)$$

$$p(i',j';1-\alpha,1-\alpha|i,j;\alpha,1-\alpha) = \gamma_{ii'}p_\alpha(i',j'|i,j) \qquad (C1.8d)$$

Case 4. Both players apply an aspiration test. This occurs with probability λ^2. The possible non-zero transition probabilities are

$$p(i',j';\alpha,\alpha|i,j;\alpha,\alpha) = (1-\gamma_{ii'})(1-\gamma_{jj'})\bar{p}_{\omega}(i',j'|i,j) \quad (C1.9a1)$$

$$p(i',j';\alpha,1-\alpha|i,j;\alpha,\alpha) = (1-\gamma_{ii'})\gamma_{jj'}\bar{p}_{\omega}(i',j'|i,j) \quad (C1.9a2)$$

$$p(i',j';1-\alpha,\alpha|i,j;\alpha,\alpha) = \gamma_{ii'}(1-\gamma_{jj'})\bar{p}_{\omega}(i',j'|i,j) \quad (C1.9a3)$$

$$p(i',j';1-\alpha,1-\alpha|i,j;\alpha,\alpha) = \gamma_{ii'}\gamma_{jj'}\bar{p}_{\omega}(i',j'|i,j) \quad (C1.9a4)$$

$$p(i',j';\alpha,1-\alpha|i,j;\alpha,1-\alpha) = (1-\gamma_{ii'})(1-\gamma_{jj'})p_{\omega}(i',j'|i,j) \quad (C1.9b1)$$

$$p(i',j';\alpha,\alpha|i,j;\alpha,1-\alpha) = (1-\gamma_{ii'})\gamma_{jj'}p_{\omega}(i',j'|i,j) \quad (C1.9b2)$$

$$p(i',j';1-\alpha,1-\alpha|i,j;\alpha,1-\alpha) = \gamma_{ii'}(1-\gamma_{jj'})p_{\omega}(i',j'|i,j) \quad (C1.9b3)$$

$$p(i',j';1-\alpha,\alpha|i,j;\alpha,1-\alpha) = \gamma_{ii'}\gamma_{jj'}p_{\omega}(i',j'|i,j) \quad (C1.9b4)$$

We can now obtain the *unconditional* transition probabilities (C1.3). Thus,

$$p(i',j';\alpha,\alpha|i,j;\alpha,\alpha) = \lambda^2(1-\gamma_{ii'})(1-\gamma_{jj'})\bar{p}_{\omega}(i',j'|i,j) \quad (C1.10a1)$$

$$p(i',j';\alpha,1-\alpha|i,j;\alpha,\alpha) = \lambda(1-\gamma_{ii'})\big(1-\lambda(1-\gamma_{jj'})\big)\bar{p}_{\omega}(i',j'|i,j) \quad (C1.10a2)$$

$$p(i',j';1-\alpha,\alpha|i,j;\alpha,\alpha) = \big(1-\lambda(1-\gamma_{ii'})\big)\lambda(1-\gamma_{jj'})\bar{p}_{\omega}(i',j'|i,j) \quad (C1.10a3)$$

$$p(i',j';1-\alpha,1-\alpha|i,j;\alpha,\alpha) = \big(1-\lambda(1-\gamma_{ii'})\big)\big(1-\lambda(1-\gamma_{jj'})\big)\bar{p}_{\omega}(i',j'|i,j) \quad (C1.10a4)$$

$$p(i',j';\alpha,\alpha|i,j;\alpha,1-\alpha) = (1-\lambda\gamma_{ii'})\lambda\gamma_{jj'}p_{\omega}(i',j'|i,j) \quad (C1.10b1)$$

$$p(i',j';\alpha,1-\alpha|i,j;\alpha,1-\alpha) = (1-\lambda\gamma_{ii'})(1-\lambda\gamma_{jj'})p_{\omega}(i',j'|i,j) \quad (C1.10b2)$$

$$p(i',j';1-\alpha,\alpha|i,j;\alpha,1-\alpha) = \lambda^2\gamma_{ii'}\gamma_{jj'}p_{\omega}(i',j'|i,j) \quad (C1.10b3)$$

$$p(i',j';1-\alpha,1-\alpha|i,j;\alpha,1-\alpha) = \lambda\gamma_{ii'}(1-\lambda\gamma_{jj'})p_{\omega}(i',j'|i,j) \quad (C1.10b4)$$

This completes the derivation of the transition probabilities.

Appendix D: Ergodicity results

D1. Proof of Proposition 4.1. We begin by computing the possible coefficients, $c^{ij}(i',j') = (c_1^{ij}(i',j'), c_2^{ij}(i',j'), c_3^{ij}(i',j'), c_4^{ij}(i',j'))$. By considering all the possible transitions, $(i,j) \to (i',j')$, and using the transition probabilities (4.1), it is straightforward to show:

$$c^{11} = \left\{ \begin{array}{lcl} c^{11}(1,1) & = & (0,0,0,0) \\ c^{11}(2,4) & = & (-2,1,0,1) \\ c^{11}(3,3) & = & (-2,0,2,0) \\ c^{11}(4,2) & = & (-2,1,0,1) \end{array} \right\} \text{ each with probability } \frac{1}{4} \qquad (D1.1a)$$

$$c^{22} = \left\{ \begin{array}{lcl} c^{22}(1,1) & = & (2,-2,0,0) \\ c^{22}(2,4) & = & (0,-1,0,1) \\ c^{22}(3,3) & = & (0,-2,2,0) \\ c^{22}(4,2) & = & (0,-1,0,1) \end{array} \right\} \text{ each with probability } \frac{1}{4} \qquad (D1.1b)$$

$$c^{33} = \left\{ \begin{array}{lcl} c^{33}(1,1) & = & (2,0,-2,0) \\ c^{33}(2,4) & = & (0,1,-2,1) \\ c^{33}(3,3) & = & (0,0,0,0) \\ c^{33}(4,2) & = & (0,1,-2,1) \end{array} \right\} \text{ each with probability } \frac{1}{4} \qquad (D1.1c)$$

$$c^{44} = \left\{ \begin{array}{lcl} c^{44}(1,1) & = & (2,0,0,-2) \\ c^{44}(2,4) & = & (0,1,0,-1) \\ c^{44}(3,3) & = & (0,0,2,-2) \\ c^{44}(4,2) & = & (0,1,0,-1) \end{array} \right\} \text{ each with probability } \frac{1}{4} \qquad (D1.1d)$$

and, for $i \neq j$, $c^{ij} = c^{ij}(2,4) = c^{ij}(4,2)$, with probability $p(2,4|i,j) + p(4,2|i,j) = 1$, using (3.1b) and the fact that $k(i,j) + k(j,i) = 1$. Thus,

$$\left. \begin{array}{lclcl} c^{12} & = & c^{21} & = & (-1,0,0,1) \\ c^{13} & = & c^{31} & = & (-1,1,-1,1) \\ c^{14} & = & c^{41} & = & (-1,1,0,0) \\ c^{23} & = & c^{32} & = & (0,0,-1,1) \\ c^{24} & = & c^{42} & = & (0,0,0,0) \\ c^{34} & = & c^{43} & = & (0,1,-1,0) \end{array} \right\} \text{ each with probability } 1. \qquad (D1.2)$$

Note that, since each one-step transition on Ω_N^4 is of the form $\mathbf{x} \to \mathbf{x} + c^{ij}\Delta x$, it follows from the above calculations that the corresponding state transition probabilities are independent of the custom k. Thus, the Markov process is independent of k.

Let $\mathbf{e}_1 = (1,0,0,-1)$, $\mathbf{e}_2 = (0,1,0,-1)$, and $\mathbf{e}_3 = (0,0,1,-1)$ be vectors in $\mathbb{R}^4$, and let $\Delta x = 1/N$. Then, for $\mathbf{x} \in \Omega_N^4$, we may define the *set of nearest neighbours of* $\mathbf{x}$ in Ω_N^4 to be

$$\mathcal{N}(\mathbf{x}) = \{\mathbf{x} \pm \Delta x \mathbf{e}_r \mid 1 \leq r \leq 3\} \cap \Omega_N^4 \qquad (D1.3)$$

Let $\text{int}\,\Delta^3$ be the interior of Δ^3, and $\text{int}\,\Omega^3_N = \text{int}\,\Delta^3 \cap \Omega^3_N$. Then, for $\mathbf{x} \in \text{int}\,\Omega^3_N$, set

$$\text{int}\,\mathcal{N}(\mathbf{x}) = \mathcal{N}(\mathbf{x}) \cap \text{int}\,\Omega^3_N \tag{D1.3}$$

Let $\partial\Delta^3$ be the boundary of Δ^3 and $\partial\Omega^3_N = \partial\Delta^3 \cap \Omega^3_N$. In order to prove Proposition A, it suffices to show that, for each $\mathbf{x} \in \Omega^3_N$, there are paths of finite length and positive probability in Ω^3_N joining $\mathbf{x}$ to each point of $\text{int}\,\mathcal{N}(\mathbf{x})$. From this it follows that there is a finite, positive probability path joining any two points in $\text{int}\,\Omega^3_N$, and that any point on $\partial\Omega^3_N$ eventually moves into $\text{int}\,\Omega^3_N$.

Let $\mathbf{x} \in \Omega^3_N \subset \Delta^3$. Then the single step transition, $\mathbf{x} \to \mathbf{x} + \epsilon^{ij}(i', j')\Delta x$, occurs with probability $p(i', j' | i, j) f_{ij}(\mathbf{x}, \Delta x)$, where

$$f_{ij}(\mathbf{x}, \Delta x) = \frac{x_i(x_j - \delta_{ij}\Delta x)}{1 - \Delta x}$$

is the without replacement probability of choosing players of classes i and j. We first show that, for $\mathbf{x} \in \text{int}\,\Omega^3_N$,there is a finite sequence of single step transitions, each of positive probability, linking $\mathbf{x}$ to $\mathbf{x} \pm \Delta x \mathbf{e}_1$, whenever this latter point is also in $\text{int}\,\Omega^3_N$.

Since $\mathbf{x} \in \text{int}\,\Omega^3_N$, we have that $x_i \geq \Delta x$ for each i. Such points exist since $N \geq 3$. Then, using (D1.2) and (D1.1b), a possible path is

$$\mathbf{x} \xrightarrow{1\times 2} \mathbf{x} - \Delta x \mathbf{e}_1 \xrightarrow{3\times 3} \mathbf{x} + \Delta x \mathbf{e}_1 \tag{D1.5}$$

where $\xrightarrow{i\times j}$ means that thetransition isdue toagamebetweenplayers ofclasses i and j. Here, the first transition occurs with positive probability $C x_1 x_2$, and the second with positive probability $\frac{1}{3} C(x_3 + \Delta x)x_3$, where $C = 1/(1 - \Delta x)$. Similarly, a possible path is

$$\mathbf{x} \xrightarrow{1\times 3} \mathbf{x} - \Delta x \mathbf{e}_1 + \Delta x \mathbf{e}_2 \xrightarrow{2\times 2} \mathbf{x} - \Delta x \mathbf{e}_1 \tag{D1.6}$$

with positive step probabilities $C x_1 x_3$ and $\frac{1}{2} C(x_2 + \Delta x)x_2$.

The other nearest neighbour transitions may be effected as follows. If $x_3 \geq 2\Delta x$,

$$\mathbf{x} \xrightarrow{3\times 3} \mathbf{x} + \Delta x \mathbf{e}_2 \tag{D1.7}$$

with positive step probability $\frac{1}{2}Cx_4(x_4-\Delta x)$. If $x_2 \geq 2\Delta x$,

$$\mathbf{x} \xrightarrow{2\times 2} \mathbf{x} - \Delta x\,\mathbf{e}_2 \tag{D1.8}$$

with positive step probability $\frac{1}{2}Cx_2(x_2-\Delta x)$. Also,

$$\mathbf{x} \xrightarrow{2\times 3} \mathbf{x} - \Delta x\,\mathbf{e}_3 \xrightarrow{4\times 4} \mathbf{x} + \Delta x\,\mathbf{e}_3 \tag{D1.9}$$

with positive step probabilities Cx_2x_3 and $\frac{1}{2}C(x_4+\Delta x)x_4$, and

$$\mathbf{x} \xrightarrow{3\times 4} \mathbf{x} + \Delta x\,\mathbf{e}_2 - \Delta x\,\mathbf{e}_3 \xrightarrow{2\times 2} \mathbf{x} - \Delta x\,\mathbf{e}_3 \tag{D1.10}$$

with positive step probabilities Cx_3x_4 and $\frac{1}{2}C(x_2+\Delta x)x_2$.

Note that the assumption $x_4 \geq 2\Delta x$ in (D1.7) is without loss of generality. For, if $x_4 \leq \Delta x$, then $\mathbf{x} + \Delta x\,\mathbf{e}_2 \notin \mathrm{int}\,\Omega_N^1$, so we don't need to effect this transition. A similar remark applies to (D1.8).

It remains to show that there is a finite, positive probability path from any $\mathbf{x} \in \delta\Omega_N^1$ into $\mathrm{int}\,\Omega_N^1$. To do this, first suppose that $N \geq 7$. Then, if $x_1 = 0$, there is at least one $j = r$ for which $x_j \geq 3\Delta x$. Thus, we can effect a transition, $(j,j) \to (r,j') \in \{(1,1),(2,4),(3,3),(4,2)\}$, with positive probability $\frac{1}{4}Cx_j(x_j-\Delta x)$. It follows that there is a positive probability transition after which x_1 increases to $x_1' \geq \Delta x$, x_j decreases to $x_j' \geq x_j - 2\Delta x \geq \Delta x$, and $x_s' \geq x_s$ for $s = r$ or j. Then $\mathbf{x}$ can be moved into $\mathrm{int}\,\Omega_N^1$ after at most 3 such transitions. In fact, by a careful consideration of the various possibilities, this result can also be shown to hold for $N = 4, 5$ and 6. We omit the laborious detatils. This completes the proof of Proposition A. □

D2. Proof of Proposition 6.1. Let $\mathbf{0} \in \mathbb{R}^4$ be the zero vector, and define vectors $\mathbf{e}_r^A = \mathbf{e}_r \times \mathbf{0}$, $\mathbf{e}_r^B = \mathbf{0} \times \mathbf{e}_r \in \mathbb{R}^8$, for $1 \leq r \leq 3$, and $\mathbf{f} = (0,0,0,-1)\times(0,0,0,1) \in \mathbb{R}^8$. Then any two points in the lattice $\Omega_N^{(4,4)} \subset \Delta^7 \subset \mathbb{R}^8$ may be joined by a sequence of elementary transitions of the form

$$\mathbf{x} \longrightarrow \mathbf{x} \pm \Delta x\,\mathbf{e}_r^A \tag{D2.1a}$$

$$\mathbf{x} \longrightarrow \mathbf{x} \pm \Delta x\,\mathbf{e}_r^B \tag{D2.1b}$$

$$\mathbf{x} \longrightarrow \mathbf{x} \pm \Delta x\,\mathbf{f} \tag{D2.1c}$$

We first show that each of the transitions (D2.1) can be effected with positive probability whenever both **x** and the terminal point lie in $\mathrm{int}\,\Omega_N^{(k_1,k_2)}$.

If $\mathbf{x} = \mathbf{x}_A \times \mathbf{x}_B \in \mathrm{int}\,\Omega_N^{(k_1,k_2)}$, then $n_{(i,A)}, n_{(i,B)} \geq \Delta n$ for each i. Hence, the transitions (D2.1a)canbeeffectedbyinteractionswithplayerswhoalsousecustom A,asin(D1.5)-(D1.10). Similarly for the transitions (D2.1b). Now suppose player-I has state $(4, A)$ and player-II has state $(4, B)$. Then, from (B1.6c), a transition $((4, A), (4, B)) \to ((2, B), (4, B))$ occurs with probability

$$p(2, 4; B, B | 4, 4; A, B) = \frac{1}{4}\lambda\gamma_{42}(1 - \lambda\gamma_{44}) = \frac{1}{6}\lambda$$

using the form (6.3) for $\gamma_{ii'}$. This is positive when $\lambda > 0$. In the notation of (D2.1), this transition is

$$\mathbf{x} \longrightarrow \mathbf{x}' = \mathbf{x} + \Delta n\mathbf{f} + \Delta n\mathbf{e}_2^B. \qquad (D2.2)$$

Now, since $n'_{(2,B)} \geq 2\Delta n$, the transition (D1.8) may be performed on $\mathbf{x}'_B$ with positive probability. Composing this with (D2.2) then effects the transition $\mathbf{x} \to \mathbf{x} + \Delta n\mathbf{f}$ with positive probability. The move $\mathbf{x} \to \mathbf{x} - \Delta n\mathbf{f}$ may be constructed in a similar manner by considering the possible transitions $((4, A), (4, B)) \to ((4, A), (2, A))$.

Finally, if $\mathbf{x} \in \delta\Omega_N^{(k_1,k_2)}$, then either $\mathbf{x}_A$ or $\mathbf{x}_B$ has a zero component. But, as discussed for the one custom case, provided $\ell_A = \sum_i n_{(i,A)} \geq 4\Delta n$, all the components of $\mathbf{x}_A$ may be made non-zero by a sequence of positive probability interactions with players who also use custom A. These interactions leave $\mathbf{x}_B$ unaffected. Similarly for the components of $\mathbf{x}_B$. If, on the other hand, $\ell_A < 4\Delta n$, then $\ell_B > 4\Delta n$ (because $N \geq 8$), and there is at least one i for which $n_{(i,B)} \geq 2\Delta n$. We may therefore increase ℓ_A and decrease ℓ_B by Δn, without reducing any component of $\mathbf{x}_B$ to zero, via a transition of the form $((i, B), (i, B)) \to ((i', A), (j', B))$, with probability

$$\sum_{i',j'} p(i', j'; A, B | i, i; B, B) =$$

$$\frac{1}{4}\lambda\left\{\gamma_{i1}(1 - \lambda\gamma_{i1}) + \gamma_{i2}(1 - \lambda\gamma_{i2}) + \gamma_{i3}(1 - \lambda\gamma_{i3}) + \gamma_{i4}(1 - \lambda\gamma_{i4})\right\}$$

(see B1.6c). With the form (6.3) for $\gamma_{ii'}$, this is positive for $0 < \lambda \leq 1$ and $0 < \eta < 1$. Thus, by a finite sequence of such manouvers, we may ensure that both ℓ_A and $\ell_B \geq 4\Delta n$.

We have now shown that any point in $\Omega_{\mathcal{F}}^{(k_1,k_2)}$ may be joined to any point in $\mathrm{int}\,\Omega_{\mathcal{F}}^{(k_1,k_2)}$ by a finite, positive probability path. This proves Proposition 6.1. □

D3. Proof of Proposition 7.1. Represent a point $\mathbf{x} \in \Omega_{\mathcal{F}} \subset \Delta^{255} \subset \mathbb{R}^{256}$ as a product, $\mathbf{x} = \Pi_k \mathbf{x}_k$, with $\mathbf{x}_k \in \mathbb{R}^4$ the state vector for custom k. Here, k runs over the full set of the 64 possible customs. Denote by $\mathbf{e}_q^k \in \mathbb{R}^{256}$ the vector with components $\mathbf{x}_{k'} = \mathbf{0}$ for $k' = k$, and $\mathbf{x}_k = \mathbf{e}_q$. Also, for a pair of customs, (k, k'), with $k = k'$, denote by $\mathbf{f}^{(k,k')}$, the vector with components

$$\mathbf{f}_m^{(k,k')} = \begin{cases} (0,0,0,-1) & \text{for } m = k \\ (0,0,0,1) & \text{for } m = k' \\ \mathbf{0} & \text{otherwise} \end{cases} \qquad (D3.1)$$

Then, since $\mathbf{f}^{(k',k)} = -\mathbf{f}^{(k,k')}$, any two points in the lattice $\Omega_{\mathcal{F}}$ may be joined by a sequence of elementary transitions of the form

$$\mathbf{x} \longrightarrow \mathbf{x} \pm \Delta a\, \mathbf{e}_q^k \qquad (D3.2a)$$

$$\mathbf{x} \longrightarrow \mathbf{x} + \Delta a\, \mathbf{f}^{(k,k')} \qquad (D3.2b)$$

for which k' differs from k by a single local modification of the form (19). We first show that each of the transitions (D3.2) can be effected with positive probability whenever both $\mathbf{x}$ and the terminal point lie in $\mathrm{int}\,\Omega_{\mathcal{F}}$.

If $\mathbf{x} \in \mathrm{int}\,\Omega_{\mathcal{F}}$, then $a_{(i,k)} \geq \Delta a$ for each (i,k). Hence, the transitions (D3.2a) can beeffected by interactions with players who also use custom k, as in (D1.5)-(D1.10).

For (D3.2b), suppose that $k' = {}^{(ij)}k$ for some $i < j$ (see (C1.2)), and set $a = k(i,j)$. Consider a transition $[(i,k'),(j,k)] \to [(i',k'),(j',k')]$. By (C1.4c), (C1.10a2) and (C1.5b), a transition of this form occurs with probability

$$P(k \to k') = \sum_{i',j'} p(i',j'; 1-a, a | i,j; 1-a, 1-a)$$
$$\lambda\Big\{(1-a)(1-\gamma_{i'i})|1-\lambda(1-\gamma_{j'i})| + a(1-\gamma_{i'j})|1-\lambda(1-\gamma_{j'j})|\Big\}.$$

Here, $a \in \{0,1\}$ and $(i,j) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$. Using the form (6.3) for $\gamma_{ii'}$, one easily checks that $P(k \to k')$ is non-zero in all these cases, provided $0 < \lambda, \mu < 1$.

Such a positive probability transition results in a state change of the form

$$\mathbf{x} \longrightarrow \mathbf{x}' = \mathbf{x} + \Delta x\,\mathbf{f}^{(k,k')} + (1-\alpha)\Delta x\left(2\mathbf{e}_i^{k'} - \mathbf{e}_l^{k'}\right) + \alpha\Delta x\left(2\mathbf{e}_j^{k'} - \mathbf{e}_l^{k'}\right) \qquad (D3.3)$$

(note that $i \leq 3$). Providing $\mathbf{x}'_{k'} \in \mathrm{int}\,\Omega^{k'}_N$ (*e.g.* if $x_{(l,k')} \geq 2\Delta x$) we may compose (D3.3) with transitions of the form (D3.2a) to effect (D3.2b) with positive probability. On the other hand, if $\mathbf{x}'_{k'}$ has a zero component, then, providing $n_{k'} = \sum_i x_{(i,k')} \geq 4\Delta x$, we may connect this point to any interior point by a positive probability path, as in the one custom case, without affecting any of the other components of $\mathbf{x}$. In particular, we may connect $\mathbf{x}'$ to the interior point $\mathbf{x} + \Delta x\,\mathbf{f}^{(k,k')}$.

It remains to show that we can arrange for any point $\mathbf{x} \in \delta\Omega_N$ to satisfy $n_k \geq 4\Delta x$ for each k. Suppose $n_{k'} < 4\Delta x$ for some k'. Then, since $N \geq 256$, there exists $k = k'$ such that $n_k > 4\Delta x$. We may then choose a sequence, $k = k_0 \to k_1 \to \ldots \to k_{q-1} \to k_q = k'$, such that k_i is obtained from k_{i-1} by a single local modification. Furthermore, we may assume that $n_{k_i} = 4\Delta x$ for $1 \leq i < q$. For, if $n_{k_i} < 4\Delta x$, we may take $k' = k_i$, and if $k_i > 4\Delta x$, we may take $k = k_i$. It suffices to show therefore, that if k' is obtained from k by a single local modification, then there is a positive probability transition having the effect $n_k \to n_k - \Delta x$ and $n_{k'} \to n_{k'} + \Delta x$.

Suppose that $k' = {}^{(i,j)}k$ with $i < j$. We may assume that $x_{(i,k)}$ and $x_{(j,k)} \geq \Delta x$. For, if not, then $n_k > 4\Delta x$ meansthat, asdescribed fortheonecustom case,apreliminary shufflingamongst thecomponentsof $\mathbf{x}_k$ caneffectthiswithpositiveprobability, whileleavingtheothercomponents of $\mathbf{x}$ unchanged. We now consider a transition of the form $((i,k),(j,k)) \to ((i',k),(j',k'))$. By (C1.4c), (C1.10b1) and (C1.5a), such a transition occurs with probability

$$\begin{aligned} P(k \to k') &= \sum_{i',j'} p(i',j';\alpha,\alpha|i,j;\alpha,1-\alpha) \\ &= \lambda\left\{(1-\alpha)(1-\lambda\gamma_{i3})\gamma_{j2} + \alpha(1-\lambda\gamma_{i2})\gamma_{j3}\right\} \\ &= \alpha\lambda\gamma_{j3} \end{aligned}$$

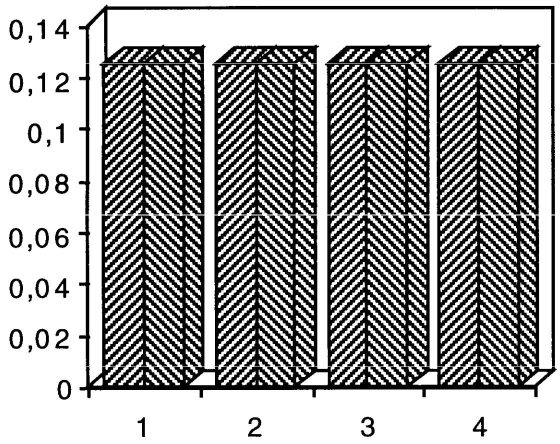
using the form (6.3) for $\gamma_{ii'}$. This is non-zero provided $\lambda, \mu > 0$ and $\alpha = k(i,j) = 1$. Similarly for transitions of theform $((i,k),(j,k)) \to ((i',k'),(j',k))$, wehave, from (C1.4b), (C1.10b4) and (C1.5a),

$$
\begin{aligned}
P(k \to k') &= \sum_{i',j'} p(i',j';1-\alpha,1-\alpha \,|\, i,j;\alpha,1-\alpha) \\
&= \lambda\left\{(1-\alpha)\gamma_{i2}(1-\lambda\gamma_{j2}) + \alpha\gamma_{i2}(1-\lambda\gamma_{j2})\right\} \\
&= (1-\alpha)\lambda\gamma_{i2}
\end{aligned}
$$

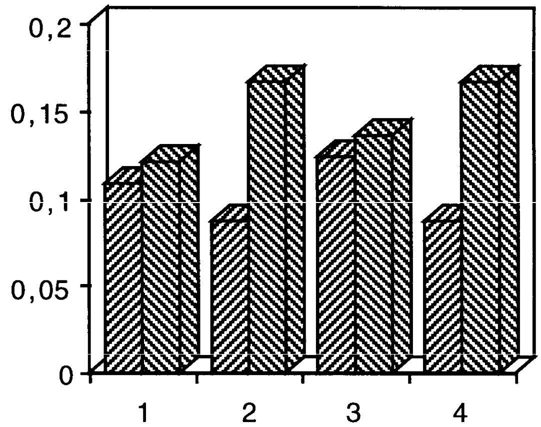
which is non-zero provided $\lambda, \gamma > 0$ and $\alpha = 0$. We have therefore completed the proof of Proposition 7.1. □

k_1 k_{64}

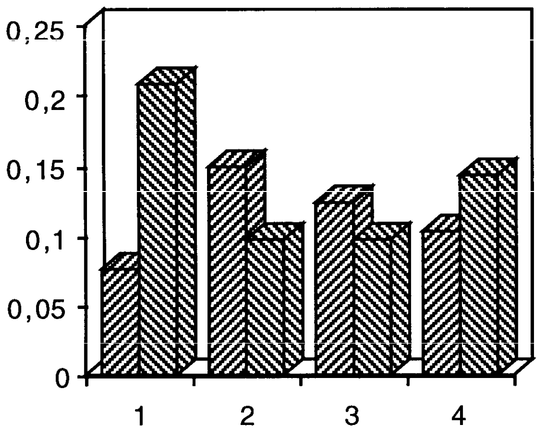
		$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
	k_1	.5	2.5	0.677
λ =**.001**	k_{64}	.5	2.5	1.661
η =**.001**	TOT	1	2.5	1.25



		$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
	k_1	.409	2.466	0.677
λ =**.001**	k_{64}	.591	2.591	1.661
η =**.999**	TOT	1	2.54	1.326



		$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
	k_1	.452	2.564	0.677
λ =**.999**	k_{64}	.548	2.319	1.661
η =**.001**	TOT	1	2.43	1,254



		$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
	k_1	.359	2.38	0.677
λ =**.999**	k_{64}	.641	2.73	1.661
η =**.999**	TOT	1	2.6	1.359

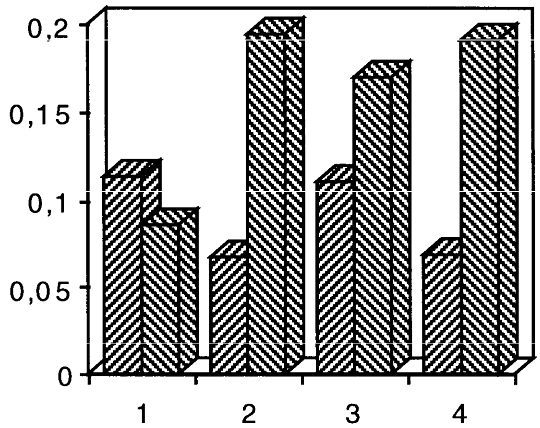


Figure 6.1

.001 - .001		WINS	x			u			B		
			av	min	max	av	min	max	av	min	max
k1	σ=3	36	0.51	0.48	0.54	2.68	2.5	2.82	0.97	0.67	1.27
	σ=4	36	0.51	0.48	0.54	2.68	2.5	2.82	0.97	0.67	1.27
k64	σ=3	27	0.49	0.46	0.51	2.68	2.5	2.82	1.46	1.250	1.65
	σ=4	27	0.49	0.465	0.51	2.68	2.5	2.82	1.462	1.25	1.65

.001 - .999		WINS	x			u			B		
			av	min	max	av	min	max	av	min	max
k1	σ=3	1	0.47	0.41	0.5	2.67	2.47	2.82	1	0.67	1.34
	σ=4	0	0.48	0.45	0.5	2.67	2.48	2.82	0.99	0.67	1.32
k64	σ=3	62	0.54	0.5	0.59	2.7	2.59	2.82	1.49	1.33	1.65
	σ=4	63	0.52	0.5	0.55	2.69	2.55	2.82	1.48	1.29	1.65

.999 - .001		WINS	x			u			B		
			av	min	max	av	min	max	av	min	max
k1	σ=3	3	0.48	0.45	0.5	2.74	2.56	2.83	0.99	0.67	1.31
	σ=4	3	0.48	0.45	0.5	2.74	2.56	2.83	0.99	0.67	1.31
k64	σ=3	36	0.5	0.45	0.55	2.61	2.32	2.81	1.45	1.23	1.65
	σ=4	36	0.5	0.45	0.55	2.61	2.32	2.81	1.45	1.23	1.65

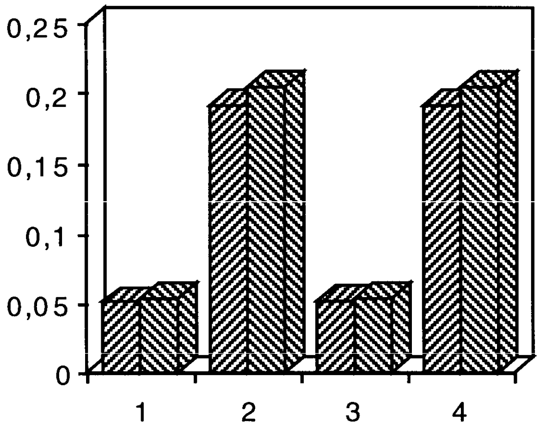
.999 - .999		WINS	x			u			B		
			av	min	max	av	min	max	av	min	max
k1	σ=3	0	0.43	0.34	0.5	2.61	2.38	2.81	1.01	0.67	1.39
	σ=4	0	0.46	0.42	0.5	2.61	2.41	2.81	0.99	0.67	1.33
k64	σ=3	60	0.55	0.49	0.64	2.74	2.63	2.82	1.5	1.35	1.65
	σ=4	62	0.52	0.5	0.57	2.74	2.647	2.82	1.48	1.3	1.65

TABLE 6.1

Legend: k_1, k_{33}

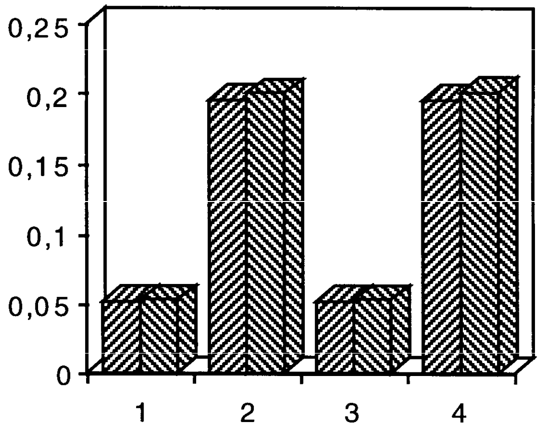
$\lambda = .001$, $\eta = .001$

	$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
k_1	.485	2.79	.667
k_{64}	.515	2.793	.667
TOT	1	2.792	.667



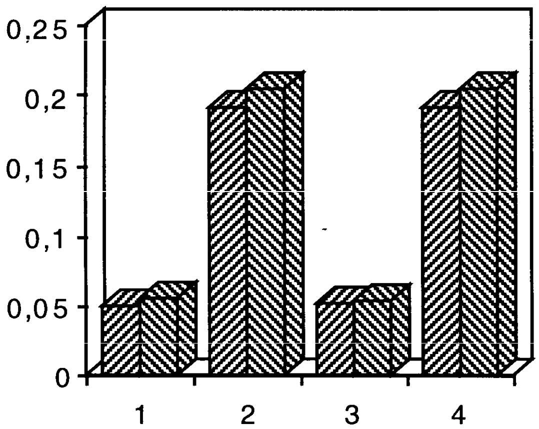
$\lambda = .001$, $\eta = .999$

	$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
k_1	.495	2.791	.667
k_{64}	.505	2.792	.667
TOT	1	2.791	.667



$\lambda = .999$, $\eta = .001$

	$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
k_1	.483	2.793	.667
k_{64}	.517	2.787	.667
TOT	1	2.79	.667



$\lambda = .999$, $\eta = .999$

	$\hat{x}$	$\hat{\pi}$	$\hat{B}(k)$
k_1	.489	2.78	.667
k_{64}	.511	2.80	.667
TOT	1	2.79	.667

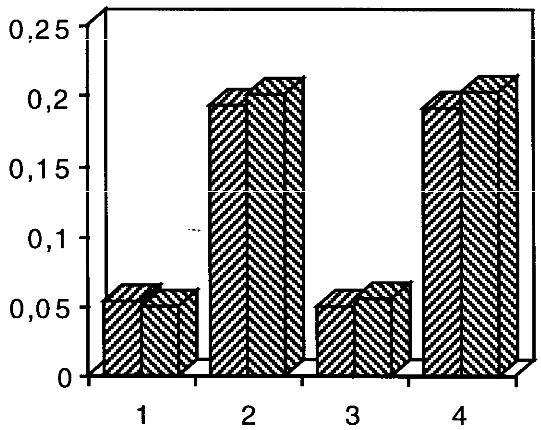


Figure 6.2

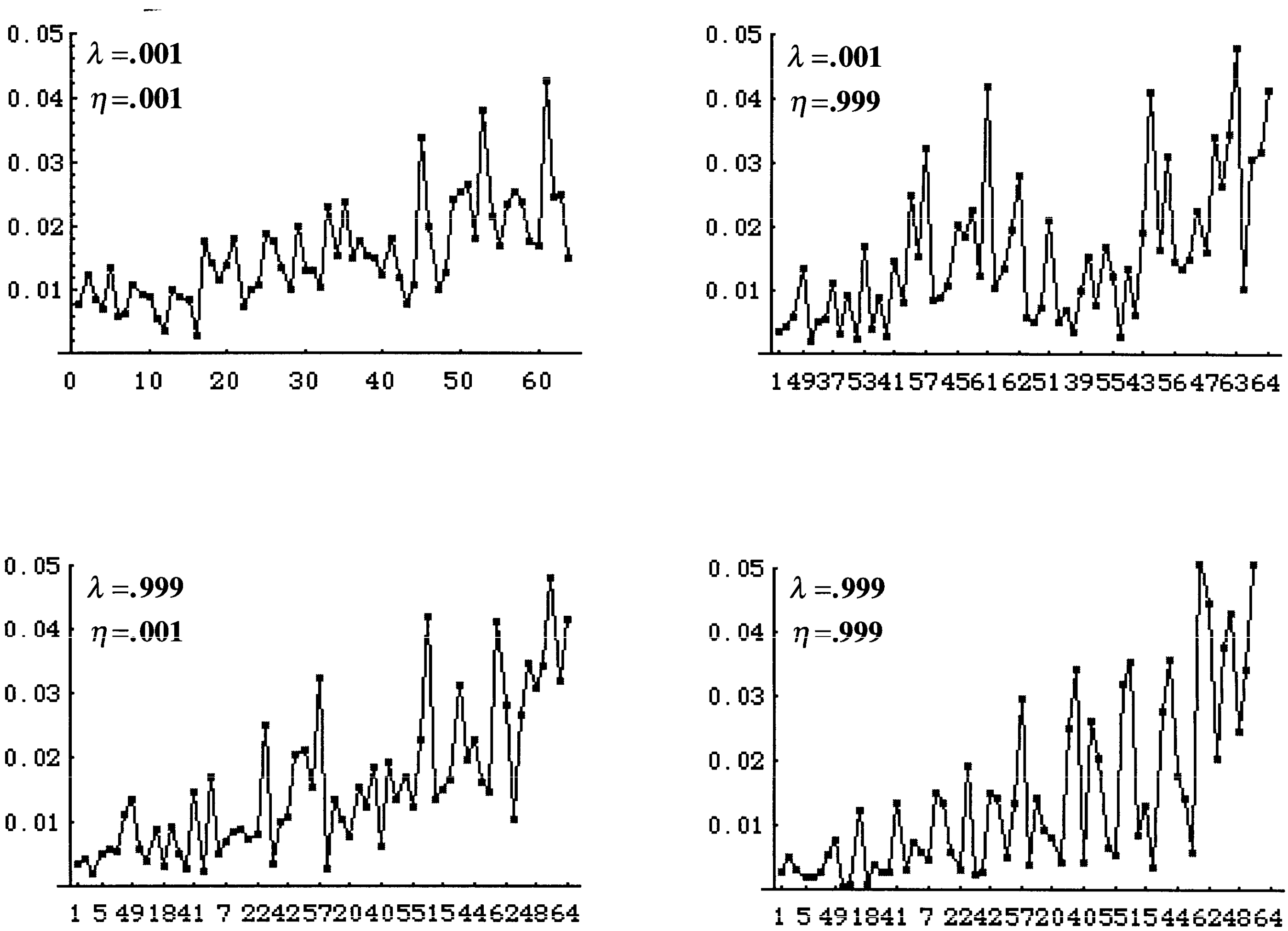

λ =.001
η =.001
λ =.001
η =.999
λ =.999
η =.001
λ =.999
η =.999
0.05
0.04
0.03
0.02
0.01
0 10 20 30 40 50 60
1 4937534157456162513955435647636 4
1 5 4918 41 7 2242572040551544624864

Figure 7.1