EFFICIENCY OF LARGE DOUBLE AUCTIONS

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ABSTRACT: We consider large double auctions with private values. Values need be neither symmetric nor independent. Multiple units may be owned or desired. Participation may be stochastic. We introduce a very mild notion of “a little independence.” We prove that all non-trivial equilibria of auctions that satisfy this notion are asymptotically efficient. For any $\alpha > 0$, inefficiency disappears at rate $1/n^{2-\alpha}$.

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1. Introduction

Many market settings are approximated by a double auction. Standard examples are the London gold market, and the order books maintained by NYSE specialists. These auctions typically have many traders on each side of the market.

More importantly, large double auctions are an excellent model for microfoundations of price formation in competitive markets. Like a competitive market, a large double auction has many traders. However, unlike the standard competitive model, traders are strategic. Hence, if traders asymptotically ignore their effect on price this is a result, not an assumption. And, there is an explicit mechanism translating individual behaviors into prices. So, one of the thorniest problems of the standard Walrasian model — how does the market get to equilibrium if everyone is a price taker — is explicitly addressed. Finally, double auctions are a better setting for thinking about price formation than one-sided auctions, both because they are often a better match to reality, and especially because they capture the essential problems of trade better than a one-sided auction. A large one-sided auction allows one to ask if traded units end up in the right hands. But, it does not address whether the correct number of units trade in the first place.

In a seminal paper, Rustichini, Satterthwaite and Williams (1994, henceforth RSW) consider a double auction in which \( n \) buyers and sellers draw private values iid. They show that symmetric, increasing, differentiable equilibria in this setting are in the limit efficient and that convergence is fast, of order \( 1/n^2 \). This is especially attractive in light of experimental evidence on efficiency in double auctions with only a moderate number of players.\(^2\)

In independent work, Fudenberg, Mobius and Seidel (2003) extend RSW to a setting in which a one dimensional state is sampled and values are then drawn iid from a density that depends on the state, but has non-shifting support and uniform lower bound across states.\(^3\) They also show existence of a pure increasing symmetric equilibrium when the number of players is large.\(^4\)

\(^2\)Satterthwaite and Williams (2002) establish that in the iid setting, this rate is fastest among all mechanisms. Important precursors to RSW include Chatterjee and Samuelson (1983), Wilson (1985), Gresik and Satterthwaite (1989), and Satterthwaite and Williams, (1989).

\(^3\)We encompass this case. See Example 4 below.

\(^4\)Jackson and Swinkels (2001) shows existence of non-trivial equilibria in double auctions. FMS shows that in the setting they consider, one of these equilibria is pure and increasing.
These results are useful in thinking about how auctions approximate competitive equilibria. However, there are several dimensions along which they could be strengthened.

1. The proof technique depends heavily on symmetric distributions of values.

2. Even in the symmetric setting, there is no guarantee of uniqueness. So, while well-behaved symmetric equilibria are asymptotically efficient, there may be other (possibly asymmetric) equilibria as well. In particular, there is always the no-trade equilibrium in which all buyers make an offer of zero, and all sellers make an offer higher than any possible valuation. Results before this paper do not rule out other intermediate trade equilibria.

3. While one may be willing to rule out the asymmetric equilibria on a priori grounds in the symmetric case, selecting the “good” equilibria is much harder if the initial setting is itself asymmetric.

4. Imposing symmetry on values and bids assumes away half the problem. Objects that trade automatically move from and to the right people, and so the only question is whether the volume of trade is right. Without symmetry, it may also occur that, for example, a low-valued buyer wins an object when a higher-valued buyer does not.

5. Finally, these papers consider only single unit demands and supplies.

We present a model and results addressing these points. We consider a generalized private value double auction setting. Players can be highly asymmetric, and demand or supply multiple units. Beyond the assumption of private values, there are only three assumptions with any bite. First, while individual values need be neither full support or even non-degenerate, we require that any given interval in the support of values is eventually hit in expectation by many players. We term this condition no asymptotic gaps.

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\footnote{A beautiful paper by Perry and Reny (2003) extends the previous work on information aggregation in large one sided common value auctions (Wilson 1977, Milgrom 1979, Psenderorfer and Swinkels 1997, etc.) to the double auction setting. A symmetric single unit demand and supply setting is maintained. Using a discrete bid space to get existence, they show that in a one dimensional affiliated setting, the equilibrium price converges to the rational expectations equilibrium value. So, Perry and Reny generalizes RSW in the direction of non-private values while retaining most of other restrictions, while we generalize RSW in most other directions, while, critically, retaining private values.}
(NAG). Analogously, we require there to be no asymptotic atoms (NAA): it cannot be the case that a positive limiting fraction of players are expected to pile up in an arbitrarily small interval.

Most critically, we drastically relax independence. We require only that a “little” independence across players persists as the number of players grows. A sequence of distributions over player values satisfies $z$-independence, $z \in (0, 1]$, if the probability of any given event on player $i$’s values changes by a factor bounded between $z$ and $1/z$ when one conditions on the values of the remaining players. The key requirement is that $z$ holds uniformly in the number of players. Two perfectly correlated random variables do not satisfy $z$-independence for any $z > 0$; $1$-independence is the standard notion of independence.

An interpretation of $z$-independence is that each player has at least a small idiosyncratic component to his valuation, one that cannot be precisely predicted no matter how much one knows about the values of other players. This is a weak condition, admitting very broad classes of distributions.

Because values can be highly correlated (positively, negatively or otherwise) under $z$-independence, even in the limit the allocation and price setting problem will generally be non-trivial.

There is always a no-trade equilibrium in a double auction setting. Jackson and Swinkels (2001, henceforth JS) show that there is at least one non-trivial equilibrium as well. But, because our setting is so general, these equilibria need not be in increasing or even pure strategies. Despite this, our major result is simply stated:

*As the number of players grows, every non-trivial equilibrium of the double auction setting converges to the Walrasian outcome.*

*Inefficiency disappears at rate $1/n^{2-\alpha}$ for any $\alpha > 0$.*

Asymptotic efficiency implies asymptotic uniqueness and pureness: over relevant ranges, bids must be arbitrarily close to value. Thus, as $n$ grows large, there are precisely two types of equilibria of private value double auctions:

1. equilibria involving no trade

2. equilibria in which a near efficient level of trade occurs, at a price near the competitive one.

With single unit demands and supplies, our proof works because in each outcome of a double auction, there is at most one buyer who is both currently
winning an object and who would have raised the price had he bid more (the
lowest winning buyer). So, while many buyers might have raised price by
bidding more, only one would care that he did so. This is symmetric for
sellers considering lowering their bids. So, the expected relevant impact on
price from increased bids by buyers is already order $1/n$. And, even if an
increase in a bid increases price, it should do so by an amount related to
$1/n$, since this should be the expected distance to the next bid. But then,
since the expected impact on price is order $1/n^2$, it must be that bidding
honestly almost never wins an extra object, and so those objects that are
traded must be allocated very efficiently.

The focus then turns to showing that the right number of objects trade,
or, equivalently, that the competitive gap defining the range of market clear-
ing prices grows small. This turns out to be much the hardest part of the
paper (especially with a rate). In the symmetric case, one can appeal to the
first order conditions of players near a discontinuity in bids. Here, things
are much more difficult, as without symmetric increasing strategies, (a) the
very concept of a “gap” becomes more complicated (b) it is hard to iden-
tify which player types might bid near a gap, and (c) players can have very
different beliefs about the likelihoods of the events involved. We show that
the only way to have a significant competitive gap without violating the effi-
ciency already shown for those objects traded is for the market to essentially
become deterministic, with a given set of buyers and sellers always trading.
But then, any member of either of these groups can favorably influence the
price without losing the chance to trade.

The efficiency result generalizes to multiple unit demands as long as NAG
continues to hold for the first unit of demand and supply for each player. If
this holds, we can reformulate the arguments just outlined but applied only
to the highest bid by each buyer and lowest bid by each seller to show small
price impacts of honest bidding. From there to (fast) efficiency for all units
involves a careful tracking of incentives, but is otherwise straightforward.

We begin by setting up the basic single unit demand and supply model.
We then introduce $z$-independence. Analysis of efficiency for the large double
auction with single unit demands and supplies follows. Then, we generalize
to auctions with multiple unit demands and supplies. We conclude with
some thoughts on extensions. All proofs are relegated to an appendix.

2. The Model

We begin with the structure of a given double auction $\mathcal{A}$. A finite set $N$
of players is divided into subsets $N_S$ and $N_B$. Players in $N_S$ are potential
sellers, each with one unit to sell. Players in $N_B$ are potential buyers, each desiring a single unit.

Each $i \in N$ has valuation $v_i$. For sellers, this might be either a production cost or a value in use. For $i \in N_B$, we assume $v_i \in [0,1)$. For $i \in N_S$, we assume $v_i \in (0,1]$. Ruling out buyers with value 1 and sellers with value 0 implies that a buyer with value 0 or a seller with value of 1 will never trade. Hence, one can “park” extra buyers at 0 and extra sellers at 1. There is no loss of generality in assuming an equal number of buyers and sellers.

Let $n \equiv |N_s| = |N_B|$. Because extra buyers and sellers can be parked, the model also allows a stochastic (but bounded) number of buyers and sellers. The vector $v \equiv \{v_i\}_{i \in N}$ is drawn according to a Borel probability measure $P^n$ on $[0,1)^n \times (0,1]^n$. The marginal of $P^n$ onto $v_i$ is $P^n_i$ and the marginal onto $v_{-i}$ is $P^n_{-i}$.

Throughout the paper, for any non-empty $K \subset N$, when we write $F_K$ (respectively $F_i$, $F_{-i}$, $F_{N\setminus K}$), we mean an arbitrary positive probability Borel event involving only the values or bids of the players in $K$ (respectively $\{i\}, N\setminus i, N\setminus K$). For events $F_i \subset [0,1]$ and $F_{-i} \subset [0,1]^{2n-1}$ we will let $P^n_i(F_i|F_{-i})$ be the conditional on $i$’s values.6

Each player $i$ observes his value and then submits a bid $b_i \in [0,1]$. Trade is determined by crossing the submitted demand and supply curves. Call the (random) range of possible market clearing prices the competitive gap, $cg \equiv [cg, cg]$. If we let $b^j$ denote the $j^{th}$ highest bid, then a little time with the appropriate figure shows that $cg = [b^{n+1}, b^n]$.

Assumption 1: Trade takes place at price

$$p = \hat{p}(cg, cg)$$

where $\hat{p}$ is differentiable, takes values in $[cg, cg]$, and has derivatives bounded by 0 and 1.7

Imagine that the bidder who submitted $cg$ raises his bid. As long as his bid continues to define $cg$, Assumption 1 ensures that he raises the price at rate at most 1. As soon as he passes the next bid up, he ceases to affect the

6In principle all of these objects depend on $n$. We suppress the superscript wherever possible.

7This, of course, includes the standard $k$ double auction.
price. Let \(ug = b^n - 1\) be this next bid, and define the *upper supporting gap* as \(ug \equiv [cg, ug]\). Then, the maximum effect on the price is \(|ug|\). Similarly, let \(lg \equiv b^n + 2\), and define the *lower supporting gap* as \(lg \equiv [lg, cg]\). A bidder who lowers his bid ceases to affect the price as soon as \(lg\) is passed. So, \(cg\) determines the amount of choice there is in setting a market price, while \(lg\) and \(ug\) determine how closely "supported" this range is.

Each player \(i\) has a von Neumann Morgernstern (\(vNM\)); utility function \(u_i\) with slope bounded away from 0 and \(\infty\). Thus no particular structure on risk preferences is required.\(^8\)

### 2.1. Equilibrium

A set of distributional strategies \(\{\mu_i\}_{i \in N}\) (Milgrom and Weber, 1982) is an *equilibrium* if it is a Bayesian Nash equilibrium in which buyers never bid above \(v_i\), and sellers never bid below \(v_i\). So, our definition of equilibrium encompasses a weak dominance requirement. The equilibrium is *non-trivial* if there is a positive probability of trade.

We show that non-trivial equilibria are asymptotically efficient. This, of course, is a better result if such equilibria exist! Under slightly stronger conditions than we use here, JS show that this is indeed the case.\(^9\)

### 2.2. Sequences of Auctions

Consider a sequence of such auctions \(\{A^n\}\), where \(n\) tends to infinity. We need three conditions that apply across \(n\). First, while individual values need not have full support (and may, in fact, be atomic), we require that as \(n\) grows large, each subinterval is hit with non-vanishing probability.

**Assumption 2 (No Asymptotic Gaps):** There is \(w > 0\) such that for all \(n\), and for all intervals \(I \subseteq (0, 1)\) longer than \(1/n\),

\[
\sum_{i \in N_B} P_i^n[I] \geq wn |I|
\]

\(^8\)In the proofs, we assume risk neutrality. Dealing with \(vNM\) utility functions with slope bounded away from 0 and \(\infty\) involves scaling potential gains and losses by some factor from the risk neutral case. This merely introduces notation.

\(^9\)The two key assumptions are that \(P\) have a Radon-Nikodym derivative with respect to \(\Pi_i P_i\) that is bounded away from 0 and \(\infty\) and that each \(P_i\) is atomless. Neither assumption plays any further role in the development here.
and
\[ \sum_{i \in N_S} P^n_i[I] \geq wn |I| . \]

Our second assumption similarly requires not too many values fall in a given interval.

**Assumption 3 (No Asymptotic Atoms):** There is \( W < \infty \) such that for all \( n \), and for all intervals \( I \subseteq (0,1) \) longer than \( 1/n \),
\[ \sum_{i \in N_B} P^n_i[I] \leq Wn |I| \]
and
\[ \sum_{i \in N_S} P^n_i[I] \leq Wn |I| . \]

These conditions hold only on \((0,1)\), allowing a positive mass of buyers with value 0 or sellers with value 1, consistent with our earlier discussion of “parking” extra players.

**Example 1:** Let sellers \( i \in \{1, \ldots, n\} \) have \( v_i \equiv i/n \) and similarly for buyers. NAG and NAA are satisfied for \( w = W = 1 \). So, individual values need neither have full support nor be non-atomic.

Each of these two assumptions has an analog in RSW. NAG and NAA are needed for a rate of convergence result, but not for convergence itself (see Section 6).

3. **\( z \)-Independence**

Our final condition is the most important. We wish to relax independence considerably while still requiring “some persistent independence” as the population grows.

We require that knowledge about the values of players other than \( i \) provides at most a finite likelihood ratio on the value of player \( i \), independent of how many other players there are.

**Definition 1:** The sequence of probability measures \( \{P^n\} \) satisfies \( z \)-independence, \( z \in (0,1] \), if for every \( n \), for all \( i \in N \), for any positive probability events
$F_{-i}$, $F'_{-i}$ involving only $v_{-i}$ and any positive probability event $F_i$ involving only $v_i$,

$$z \leq \frac{P^n(F_i|F_{-i})}{P^n(F_i|F'_{-i})} \leq \frac{1}{z}.$$  

That is, there is still some idiosyncrasy in each $v_i$ even as the market becomes large.10

The real content of $z$-independence is in the uniformity of $z$ across $n$. For fixed $n$, $z$-independence is stronger than mutual absolute continuity of $P^n$ with respect to the product measure $\prod_{i \in N} P^n_i$ (consider a setting where, depending on $v_1$, $v_2$ is distributed according to either a uniform or a triangular distribution on $[0,1]$) but weaker than having a continuous Radon-Nikodym derivative bounded from 0 and $\infty$.11

**Assumption 4 (z-independence): There exists $z \in (0,1]$ such that $\{P^n\}$ is $z$-independent.**

### 3.1. Examples

We begin with an example that fails $z$-independence.

**Example 2:** With probability $1/2$, values are drawn iid uniform $[0,1]$, and with probability $1/2$, $x$ is drawn uniformly from $[1/n, 1-1/n]$, and values are drawn iid uniform $[x-1/n, x+1/n]$. For each $n$, $P^n$ is absolutely continuous with respect to $\prod P_i$ (and, the example is easily modified such that the Radon-Nikodym derivative is continuous as well). But, as $n \to \infty$, seeing the values of any two given players within $2/n$ of each other makes it arbitrarily likely that any given remaining player will also have such a value, and so $z$-independence is not-satisfied.

Next are several examples satisfying $z$-independence. The first illustrates the importance of applying $z$-independence one player at a time. Even under $z$-independence, a summary statistic about a large group of players can still be arbitrarily informative about a summary statistic about the rest of the players.

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10 A contemporaneous paper by Peters and Severinov (2002) uses a similar condition (in a different model) in a finite type setting. This condition is related to the notion of $\psi$-mixing described in Bradley (2005).

11 An equivalent definition of $z$-independence is that for every $i$ and $n$, the Radon-Nikodym derivative of $P^n$ w.r.t. $P^n_i \times P^n_{-i}$ exists and is essentially bounded above by $1/z$ and below by $z$. 

9
Example 3: Nature chooses \( x \in \{L, H\} \) equiprobably. If \( L \) is drawn, values are drawn iid according to density \( f(v) = 1/2 + v \). If \( H \) is drawn, values are drawn iid according to density \( f(v) = 3/2 - v \).

Let \( F_O \) be the event that less than 50% of the odd numbered buyers have value below \( 1/2 \), and let \( F_E \) be the event that less than 50% of the even numbered buyers have value below \( 1/2 \). Then, as \( n \to \infty \), \( \Pr(F_O \cap F_E^c) \to 0 \). But, under the product of the marginals, \( \prod_{i \in \mathbb{N}} P^n_i(F_0 \cap F_E^c) = \frac{1}{4} \). In particular then, the Radon-Nikodym derivative of \( P^n \) with respect to \( \prod_{i \in \mathbb{N}} P^n_i \) grows arbitrarily large as \( n \) increases.

However, \( \frac{1}{2} \)-independence is satisfied for this example: all one can extract from \( v_{-i} \) is information about whether \( x \) is \( L \) or \( H \), which changes the density on \( v_i \) from 1 to something between \( 1/2 \) and \( 3/2 \).

Example 3 generalizes to any process in which a state is sampled and then, conditional on the state, values \( v_i \) are drawn independently from measures with non-moving support \( V_i \) according to densities uniformly bounded (across states and \( n \)) away from zero and infinity. So our setting encompasses Fudenberg et al. (2003) (and more importantly, non-symmetric analogues to their model).

Postlewaite and Schmeidler (1986) define non-exclusivity as a situation where the information of \( n - 1 \) players is enough to predict the relevant state of the economy. A variety of follow-on papers relax this to hold only asymptotically.\(^{12}\) On first view, \( z \)-independence is antithetical to non-exclusivity, since no matter how much is known about the rest of the players, the value of player \( i \) remains uncertain. However, note that non-exclusivity refers to information about the underlying state, not to the signals players realize conditional on those states. In Example 3, \( v_{-i} \) is asymptotically fully informative about \( L \) vs. \( H \), while of bounded informativeness about \( v_i \). Hence Example 3 satisfies both conditions.

Example 4: Nature draws \( v_1 \) uniformly from \([0, 1]\) (this person is a “fashion leader”), and then draws subsequent players iid according to a density with support \([0, 1]\) but concentrated around \( v_1 \).

Since the impact of an early draw on later draws does not vanish, \( z \)-independence does not imply weak mixing. It is also easy to construct sequences satisfying weak mixing under which successive draws are arbitrarily correlated, violating \( z \)-independence.

Example 5: A parameter \( x \) is chosen from \([0, 1]\). Values are drawn conditionally independently according to \( f(., | x) \), where \( f(., | x) \) satisfies MLRP in

\(^{12}\) A good entry point is McLean and Postlewaite (2002).
As long as \( f(.|0)/f(.|1) \) is uniformly bounded, \( z\)-independence is satisfied for \( z = \min_x f(x|0)/f(x|1) \). Choose a subset of the players, and replace \( v_i \) by \( 1 - v_i \). This measure continues to satisfy \( z\)-independence, but is obviously not affiliated. So, affiliation has essentially nothing to do with the issues at hand.

We close this subsection with an example illustrating the surprising degree of correlation \( z\)-independence permits. Define \([x]\) as the largest integer smaller than \( x\).

**Example 6:** For \( m \leq n \), let \( \zeta_B(m) = \binom{n}{m} (0.5)^n \) be the probability of \( m \) heads from flipping \( n \) fair coins.\(^{13}\)

Now, for some \( 0 < a < 1/2 \), generate \( \zeta_C \) from \( \zeta_B \) by first defining \( \zeta_C(m) = \zeta_B(m)a^{m-[n/2]} \), and then defining \( \zeta_C \) from \( \zeta_C' \) by normalizing. Informally one makes each outcome successively further away from \([n/2]\) more unlikely by a factor of \( a \). Choose \( m \) according to \( \zeta_C \), choose each subset of coins of size \( m \) with equal probability, and make the coins in the subset heads, and the remainder tails. When \( a \) is small, drawing exactly \([n/2]\) heads by this process becomes very probable.\(^{14}\) For \( a = .1 \), e.g., there is an 80% chance of exactly \([n/2]\) heads regardless of \( n \).\(^{15}\)

None-the-less this process satisfies \( a^2 \)-independence. If there are \( m' \) heads among all but coin \( i \), the probability that \( i \) is heads is \( \Pr(m = m' + 1)/\Pr(m = m' + 1) \). By construction, this is either \( a \) or \( 1/a \).

Under \( z\)-independence, probabilities that start in the interior of \((0,1)\) cannot be moved too far toward or away from the boundaries. But, as this example illustrates, probabilities can be moved around essentially arbitrarily otherwise.

\(^{13}\)The example can easily be extended from coins to values in the standard domain.

\(^{14}\)Note that

\[
\sum_{m=0}^{r} \zeta_C = \sum_{m=0}^{r} \zeta_B(m)a^{m-[n/2]} \leq \zeta_B(r/2) \sum_{m=0}^{r} a^{m-[n/2]} \\
\leq \zeta_B(r/2) \left( 1 + 2 \sum_{i=1}^{\infty} a^i \right) = \zeta_B(r/2) \left( 1 + \frac{2a}{1-a} \right)
\]

Thus,

\[
\zeta_C(n/2) \geq \frac{\zeta_B(n/2)}{\zeta_B(n/2) \left( 1 + \frac{2a}{1-a} \right)} = \frac{1}{ \left( 1 + \frac{2a}{1-a} \right)}
\]

As \( a \to 0 \), this tends to 1.

\(^{15}\)For \( a = .1 \), the previous expression is equal to \( \frac{1}{(1+.2)} \approx .81 \).
The techniques in Mailath and Postlewaite (1990), Al-Najjar and Smorodinsky (1997), and Swinkels (2001) all rely on there being “noise” in the sense that the exact number of players taking a specific action (say bidding above some threshold) becomes diffuse. Example 6 illustrates that those techniques do not apply here.

More strikingly, in Example 1, there is no uncertainty at all. But, 1-independence is clearly satisfied. More generally, our results apply to any fully deterministic environment satisfying NAA and NAG.

It is also worth noting that the deterministic environment is a clean example where strategies are mixed, but asymptotic efficiency obtains none-the-less.

3.2. A Preliminary Lemma

Our first lemma shows that if values are $z$-independent then so too are bids. The intuition for this is that observing a player’s bid is at most as informative as observing his type.\footnote{A related lemma appears in JS.} It also describes the implications of $z$-independence for groups of players.

**Lemma 1:** Fix a non-empty $K \subset N$. Let $a = \min \{|K|, |N\setminus K|\}$. Then for all $F_K$, and $F_{N\setminus K}$,

\[
(2) \quad z^{-a} \Pr(F_K) \geq \Pr(F_K|F_{N\setminus K}) \geq z^a \Pr(F_K).
\]

Let $X_K$ be a random variable that depends only on the values/bids of the players in $K$. Then:

\[
(3) \quad z^{-a} E(X_K) \geq E(X_K|F_{N\setminus K}) \geq z^a E(X_K).
\]

When $a$ is large, these bounds are weak; for events involving many players likelihood ratios can explode.

3.3. Large Deviations

Given $K \subset N$ and events $\{F_i\}_{i \in K}$ let $Q_K$ be the number of $F_i$ that occur. We would like to understand the properties of $Q_K$. In this section, we first relate the stochastic behavior of $Q_K$ to a set of independent coins. Lemma 2 exploits the well understood large deviation properties of sets of independent coins to derive stochastic bounds on $Q_K$. 

\[
16 A related lemma appears in JS.
\]
Note first that for each $i$,

$$\Pr(F_i|F_{-i}) \geq z \Pr(F_i).$$

Also,

$$\Pr(F_i^c|F_{-i}) \leq \frac{1}{z} \Pr(F_i^c) = \frac{1}{z} (1 - \Pr(F_i))$$

and so

$$\Pr(F_i|F_{-i}) \geq 1 - \frac{1}{z} (1 - \Pr(F_i)).$$

Thus,

$$\Pr(F_i|F_{-i}) \geq p_i \equiv \max \left\{ \frac{1}{z} \Pr(F_i), 1 - \frac{1}{z} (1 - \Pr(F_i)) \right\}.$$ 

Since this is true for all $F_{-i}$, we will show that for any given $F_{N\setminus K}$, $Q_K$ first order stochastically dominates (FOSD) $|K|$ independent coins with parameters $p_i$.

Similarly, for any given $F_{N\setminus K}$, $|K|$ independent coins with parameters

$$\overline{p}_i \equiv \min \left\{ \frac{1}{z} \Pr(F_i), 1 - z(1 - \Pr(F_i)) \right\}$$

stochastically dominate $Q_K$.

We then have the following application of large deviations:

**Lemma 2**: For all $K \subset N$ and $F_{N\setminus K}$,

$$\Pr \left( Q_K < \frac{z}{3} E(Q_K) \bigg| F_{N\setminus K} \right) \leq e^{-3zE(Q_K)}$$

$$\Pr \left( Q_K > \frac{3}{z} E(Q_K) \bigg| F_{N\setminus K} \right) \leq e^{-E(Q_K)}$$

The logic underlying Lemma 2 also implies that the probability of at least one success in $K$ is not drastically affected by $F_{N\setminus K}$.

**Corollary 1**:

$$\Pr(Q_K \geq 1 \mid F_{N\setminus K}) \geq (1 - e^{-z}) \Pr(Q_K \geq 1).$$
3.4. Normal Realizations

We prove convergence at rate $1/n^{2-\alpha}$ for any given $\alpha > 0$. It is convenient to fix $\alpha$ now. We will need various fudge factors along the way. Choose $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that

\begin{equation}
\alpha > \alpha_1 > \alpha_2 > \alpha_3 > \alpha_4 > 2\alpha/3
\end{equation}

Let $w' \equiv \frac{w}{n}$ and $W' \equiv \frac{W}{n}$.

**Definition 2:** A realization of $v = \{v_i\}_{i \in N}$ is normal if every interval $I \subseteq (0,1)$ longer than $1/n^{1-\alpha/3}$ has between $w' |I|$ and $W' |I|$ buyers with value in that interval and between $w' |I|$ and $W' |I|$ sellers with value in that interval.

Let $\mathcal{N}$ be the event that the realization is normal. We use SL as an abbreviation for “sufficiently large.” A key implication of Lemma 2 is

**Lemma 3:** For all $n$ SL, $\Pr(\mathcal{N}) \geq 1 - 1/n^4$.

Together with NAG and NAA, Lemma 3 implies that the limiting realized true demand and supply curves are unlikely to have either vertical or flat sections (except at 0 for buyers and 1 for sellers).

3.5. Comments on the Rate of Convergence

Our major result is to show that percentage efficiency losses are asymptotically less than $1/n^{2-\alpha}$. As for all rate of convergence results, this does not say anything about small $n$. A weakness of our results is that the construction underlying normality only holds for $n$ extremely large.

At many other points in the paper, we sacrifice the size of the constant for transparency of exposition. And, to a certain extent, this same practice is reflected here. But, it is also the case that normality is very demanding. In particular, we require that even the emptiest interval has many values in it, despite the fact that these intervals are shrinking nearly at rate $1/n$.

It is in fact irrelevant whether most intervals have values in them or not. What matters is just the intervals near the competitive and supporting gaps. But, without further structure on the equilibrium or statistical setting, we don’t see any way of ruling out that the universe conspires against us in such a way that the emptiest interval always happens to be the relevant one. So, we use the rather crude technique of making even the emptiest interval full. It is an open question what else one would need to side-step this technique.
There seems to be nothing in the underlying incentives being exploited that precludes a small number being the $n$ that is 'sufficiently large' for the $1/n^{2-\alpha}$ rate to become effective. RSW supplement their rate result (where the constant is also large) with numerically solved examples. Such solutions are beyond our ability in this setting.

If one is satisfied to simply show convergence without a rate, then none of the infrastructure of normality is relevant.

4. Analysis of the Double Auction

4.1. Summing Deviations

Fix an equilibrium $\mu$ of $A^n$. Consider buyer $i$’s distributional strategy $\mu_i$. A deviation for $i$ is a measurable mapping $d_i$ from $[0, 1]^2$ to $[0, 1]$. First $i$ draws $v_i$ and $b_i$ according to $\mu_i$, but then she modifies her chosen bid according to $d_i$. Consider $d_i$ for which $b_i \leq d_i(b_i, v_i) \leq v_i \forall b_i, v_i$. That is, $i$ sometimes raises her bid, but not beyond her true value (since $\mu_i$ did not involve $i$ bidding more than her true value, this is coherent).

In any given realization, $d_i$ may have benefit $\hat{B}_i$ in that $i$ wins when he otherwise would not have, or may have cost $\hat{C}_i$ that $i$ pays more when he would have already won. To formalize this, let $p$ be price under $\mu$, and $p_d$ the price when $i$ uses $d_i$. Let $W_i$ be the event that $i$ wins with $d_i$, but not without. So, $\hat{B}_i = v_i - p_d$ when $W_i$ occurs, and 0 otherwise. Let

$$ B_i \equiv E(\hat{B}_i) = \Pr(W_i) E(v_i - p_d|W_i) $$

be the expectation of $\hat{B}_i$.

Similarly, let $O_i \subset W_i^c$ be the event that $i$ wins without $d_i$. Then $\hat{C}_i = p_d - p$ when $O_i$ occurs, and 0 otherwise. Let

$$ C_i \equiv E(\hat{C}_i) = \Pr(O_i) E(p_d - p|O_i). $$

Since $\mu_i$ is a best response, $B_i \leq C_i$. So, given such a deviation $d_i$ for each buyer,

$$ \sum_{N_B} B_i \leq \sum_{N_B} C_i. $$

Each $d_i$ is unilateral. But, there is nothing wrong with summing the incentive constraints implied.

Let $T$ be the event that trade occurs. Consider $\sum_{N_B} C_i$. Ex-post, $\hat{C}_i > 0$ only if (a) trade was occurring (event $T$) and (b) the original $b_i$ was equal to $c\bar{g}$, and uniquely so. To see this, note that when $b_i > c\bar{g}$ (or is tied at
If $b_i < \underline{c}$, increasing $b_i$ may increase $p$, but as $i$ was not originally winning, she is unhurt. So, there is at most one $i$ with $\hat{C}_i > 0$.

For sellers, the same analysis applies if bids are lowered, but not below value. We have thus established:

**Lemma 4:** For any set $\{d_i\}_{i \in N_B}$ of deviations for which $d_i(b_i, v_i) \in [b_i, v_i]$ for all $(b_i, v_i)$,

$$\sum_{N_B} C_i \leq \Pr(T) E (|ug| \mid T).$$

For any set $\{d_i\}_{i \in N_S}$, for which $d_i(b_i, v_i) \in [v_i, b_i]$ for all $(b_i, v_i)$,

$$\sum_{N_S} B_i \leq \sum_{N_S} C_i \leq \Pr(T) E (|lg| \mid T).$$

While easy to prove, this bound is powerful. Independent of the number of bidders, the total benefit of making new trades by bidding more aggressively must be small in equilibrium. Note also that Lemma 4 remains true if one replaces $T$ by any $T' \supseteq T$.

### 4.2. The Probability of Trade is Bounded from Zero

An important first step is to show that non-trivial equilibria are not “almost trivial” in the sense that trade becomes increasingly rare as $n$ grows. For each $n$, choose a non-trivial equilibrium of $\mathcal{A}^n$. Let $V$ be the number of objects traded and recall that $T = \{V \neq 0\}$. As before $V$ is a random variable, the distribution of which depends on both $n$ and the equilibrium under consideration. We suppress this in the notation.

**Proposition 1:** There is $\gamma > 0$ such that for all $n$ SL, and all non-trivial equilibria, $\Pr(T) \geq \gamma$.

For intuition, say that a buyer’s offer is serious if it is above (say) $\frac{1}{2}$, and a seller’s if it is below $\frac{1}{2}$. Assume that the probability of even one serious buy offer is some $\delta$ which is positive but close to zero, and similarly for $\underline{c}$.

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\[17\] If $\underline{c}$ is a seller’s bid, no buyer is hurt by $d_i$. 

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sellers (clearly, if there is a non-vanishing probability of a serious offer on either side, trade will not disappear). Trade occurs at most $2\delta$ of the time, since trade requires a serious offer from at least one side. Hence, by Lemma 4, the total costs to buyers (or sellers) of making more generous offers is like (has the same order as) $\delta$. But, from $z$-independence and Corollary 1, the probability of a serious offer from one side but not the other is like $(1 - \delta)\delta \equiv \delta$. When there is a serious offer on one side but not the other, a number of bidders on the other side that grows like $n$ would have benefited by deviating to trade at the serious offer. The aggregate gains are thus like $n\delta$, while costs are like $\delta$. This is a contradiction.

4.3. Small Supporting Gaps

We show next that the upper and lower supporting gaps shrink quickly. This will mean that the bounds in Lemma 4 are very powerful.

First, we show that $E(|ug|)$ (respectively $E(|lg|)$) is like $1/n$. The idea is most easily seen if for each $n$, $ug$ has constant length $x$. By Lemma 3, a number of buyers proportional to $nx$ will have $v_i$ in the top half of $ug$. At most one of these buyers is winning an object (they are not bidding above $\overline{ug}$, as bids are below value, and at most one bid below $\overline{ug}$ is filled). By raising $b_i$ to $v - x/2$ all but this player (acting unilaterally) would win an extra object and earn at least $x/2$. So, $\sum B_i \geq nx^2$ (up to some constants). But by the argument preceding Lemma 4, $\sum C_i \leq x$, since the one person who is hurt raises the price by at most $|ug|$. Thus,

$$nx^2 \leq \sum_{N_B} B_i \leq \sum_{N_B} C_i \leq x,$$

from which $x \leq \frac{1}{n}$. The actual proof has to account for the fact that $|ug|$ is stochastic, as are the number of bidders in any given interval. Formally:

**Lemma 5:** For $n$ SL,

$$E(|ug|) \leq \frac{1}{n^{1-\alpha_4}}, \quad E(|lg|) \leq \frac{1}{n^{1-\alpha_4}}.$$

Having bounded the expectation of $|ug|$, we can get a better handle on its distribution. Fix $x$, and consider $\Pr(|ug| \geq x)$. Consider the deviation in which buyers raise $b_i$ to $v_i - x/2$. When $|ug| \geq x$, then as above, a number

---

Note that this will be true even if under the original realization of bids there was no trade at all.
of buyers like \( n x \) makes gains \( x/2 \), and so \( \sum B_i = nx^2 \) (again ignoring constants). And, \( \sum C_i \leq E(|ug|) \leq 1/n \) from the first step. So

\[
Pr(|ug| \geq x) nx^2 \leq \frac{1}{n},
\]

from which \( Pr(|ug| \geq x) \leq \frac{1}{nx^2} \). Formally

**Lemma 6:** For \( n \) SL and all \( x \),

\[
Pr(|ug| \geq x) \leq \frac{1}{n^{2-\alpha_3}x^2}, \quad Pr(|lg| \geq x) \leq \frac{1}{n^{2-\alpha_3}x^2}.
\]

The preceding two Lemmas allow us to derive a tight bound on inefficiencies caused by misallocating the objects that actually trade (whether the right number of objects trade is the subject of the next section). For \( x \geq 0 \), let \( L_B(x) \) be those buyers with values above \( cg + x \) that do not receive an object, and let \( l_B(x) \equiv \#L_B(x) \). Similarly let \( L_S(x) \) be those sellers with values below \( cg - x \) who do not sell, and let \( l_S(x) \equiv \#L_S(x) \). Let

\[
Y_B(x) \equiv \sum_{i \in L_B(x)} v_i - cg, \quad Y_S(x) \equiv \sum_{i \in L_S(x)} cg - v_i.
\]

For buyers with values above \( cg + x \), this is the loss in consumer surplus compared with being able to price take at \( cg \), and analogously for sellers. Our next lemma uses Lemma 5 to show that both the number of such players and the associated loss is small. The intuition again comes from considering players bidding closer to their values.

**Lemma 7:** For \( n \) SL and for all \( x \),

\[
E(l_B(x)) \leq \frac{1}{xn^{1-\alpha_4}}, \quad E(l_S(x)) \leq \frac{1}{xn^{1-\alpha_4}}.
\]

Further

\[
E(Y_B(1/n)) \leq \frac{1}{n^{1-\alpha_3}}, \quad E(Y_S(1/n)) \leq \frac{1}{n^{1-\alpha_3}}.
\]

### 4.4. Small Competitive Gaps

Let us now turn to the competitive gap. Our key lemma turns out to be much the hardest to prove:

**Lemma 8:** For \( n \) SL and for all \( x \),

\[
Pr(|cg| \geq x) \leq \frac{1}{n^{2-\alpha_2}x^2}.
\]
To see the idea behind Lemma 8 consider the simpler situation in which there is some interval \( I = (\underline{I}, \overline{I}) \) of length \( x \) such that nobody ever bids in \( I \), and such that \( \Pr(I \subseteq cg) \) does not vanish quickly.

Imagine that we can show that this implies that \( \Pr(I \subseteq cg) \to 1 \). Then, by bidding \( \underline{I} + \varepsilon \), any buyer who used to bid above \( \overline{I} \) can still trade almost as often and force the price near \( \underline{I} \), while by bidding \( \overline{I} - \varepsilon \), any seller who used to bid below \( \underline{I} \) can still trade almost as often and force the price near \( \overline{I} \). One or the other of these must be profitable, contradicting equilibrium.

So, let us argue that \( \Pr(I \subseteq cg) \to 1 \). Since nobody ever bids in \( I \), each bid is either \textit{up} (above \( \overline{I} \)), or \textit{down} (below \( \underline{I} \)). Now, \( \{I \subseteq cg\} \) is equivalent to there being exactly \( n \) ups. If there are \( n + 1 \) ups then \( I \subseteq lg \), and if there are \( n - 1 \), then \( I \subseteq ug \). From the previous section, \( \Pr(I \subseteq ug) \) and \( \Pr(I \subseteq ug) \) do fall quickly. So, it must be that case that the probability of exactly \( n \) ups does not vanish quickly, but the probability of either \( n - 1 \) or \( n + 1 \) does vanish quickly. We show that the only way for this to occur is if the system becomes essentially deterministic, and so \( \Pr(I \subseteq cg) \to 1 \).

To see how this works, let \( p_i \) be the probability that \( i \) bids up, and \( q_i \) the probability he bids down. Order the players so that \( p_i \) is decreasing.

For any given realization run along them stopping at the player \( i \) when one counts \( n - 1 \) ups.

For \( I \subseteq cg \), we need to hit exactly one more up in the rest of the sequence. If one hits no more ups, \( I \subseteq ug \), while if one hits 2 more ups, \( I \subseteq lg \), either of which is rare by Lemma 6. But, we argue, the only way to make 1 more up likely, but neither 0 nor 2 more ups likely is for the next player to have \( p_{i+1} \) nearly 1, and for the remaining players to \textit{in aggregate} have almost no chance of even one up. Essentially, if \( p_{i+1} \) is not near one, then, since \( p_i \) is decreasing, the probability on who is the \( n^{th} \) up is spread out. But then, \( z \)-independence makes it likely that one also over or undershoots by 1. And, given that the next player is likely to hit, there must rarely be any more hits in the remaining population.

Running through the players in reverse order and counting downs, when one hits \( n - 1 \) downs, the next one must almost certainly play down, and then there must almost never be any more downs. Since both of these are true at once, in aggregate, the first \( n \) bidders almost always bid up and the remaining down. Hence, \( \Pr(I \subseteq cg) \to 1 \).

The proof is distressingly long: \( cg \) can move around, sometimes including one interval and sometimes another, players might bid not only above or below any given \( I \), but sometimes within it, and one must be careful not to double count the ways in which a population “one player away” from creating a long \( cg \) might end up creating a long supporting gap. Most importantly,
to derive the rate of convergence result, we need to consider situations in which \( I \) shrinks as \( n \) increases, and so must be careful in accounting for the value of the occasional lost trade from bidding lower in an attempt to favorably affect the price.

4.5. Efficiency

We are now ready for our main theorem:

**Theorem 1:** All non-trivial equilibria of the single unit demand/supply double auction are asymptotically efficient. Uniformly across non-trivial equilibria, efficiency losses go to zero faster than \( 1/n^{1-\alpha} \) for any given \( \alpha > 0 \). The fraction of expected surplus lost relative to a Walrasian market thus shrinks as \( 1/n^{2-\alpha} \).

For intuition, note that in Section 4.3 we showed that the efficiency loss from failing to trade objects between sellers with value below \( cg \) and buyers with values above \( cg \) is small (of order \( 1/n \)). So, the only efficiency losses to worry about are from pairs of buyers and sellers both having value in \( cg \). The loss from missing such a trade is at most \( |cg| \). And, using NAA, the number of such buyers and sellers is like \( |cg|/n \). So, the deadweight loss triangle from too little trade has area \( |cg|^2/n \). But, from Lemma 8, \( \Pr(|cg| \geq x) \leq \frac{1}{n^{1-\alpha}x^2} \), and so the expected loss here is like \( 1/n \) as well. Finally, from NAG, expected feasible surplus grows like \( n \), and so proportional losses are like \( 1/n^2 \). A formal accounting of efficiency losses is subsumed by the proof of the multiple unit case, and so omitted in the appendix.

4.6. Asymptotic Uniqueness of Equilibrium

In the space of allocations, all non-trivial equilibria converge to the Walrasian outcome. Over “relevant” ranges bids must thus converge to true values. So, if in the limit, the Walrasian price is either \( p_1 \) or \( p_2 > p_1 \), then, players with value near \( p_1 \) or \( p_2 \) must bid close to value. But it is difficult to show that, for example, a player with value well above \( p_2 \) must bid near value. A rate of convergence result for bids is thus cumbersome. Intuitively, over relevant ranges convergence should be order \( 1/n \).

5. Multiple-Unit Demands and Supplies

Assume now that each player has demand or supply for at most \( m \) units, for some fixed \( m \). For buyers, let \( v_{ih}, h \in \{1, \ldots, m\} \), be \( i \)'s incremental
value for unit $h$. For sellers, let $v_{ih}$ be the incremental cost of unit $h$. We assume $v_{ih}$ is non-increasing in $h$ for buyers and non-decreasing for sellers. Bids are (non-increasing for buyers, non-decreasing for sellers) $m$-vectors. JS applies to show existence of equilibria in this setting, subject to the same strengthenings as before.

We assume the following version of NAG.

**Assumption 5 (No Asymptotic Gaps*)**: There is $w > 0$ such that for all $n$, and for all intervals $I \subseteq (0, 1)$ of length $1/n$ or greater,

$$
\sum_{i \in \mathcal{N}_B} P_i[v_{i1} \in I] \geq wn |I|
$$

and

$$
\sum_{i \in \mathcal{N}_S} P_i[v_{i1} \in I] \geq wn |I|.
$$

That is, when $n$ is large, there are many buyers whose highest value might fall in any given interval, and many sellers whose lowest cost might fall into any given interval.\(^{20}\)

Note that $z$-independence applies only across players, and thus does not restrict the relationship of the different values of any given player. NAA is assumed to apply to all values, not just the first. So, not too many $v_{ih}$ fall in any given interval.

**Theorem 2**: With NAG *, Theorem 1 continues to hold even with multiple-unit demands and supplies.

Most of the incentive arguments rely only on the highest value unit of demand for buyers and lowest cost unit for sellers. The proof proceeds in two steps. Define $\overline{cg}$ as the $m^{th}$ bid up from $\underline{cg}$, and $\overline{ug} \equiv [\overline{cg}, \overline{ug}]$. In the appendix, we show that Lemma 6 continues to hold for this definition of $\overline{ug}$. The modification to the intuition is very small: when $\overline{ug}$ is long, there are many buyers with highest value in the top half of $\overline{ug}$. But, only $m$ of them can be winning a first object. Given this, Lemma 8 is easily extended as well. Instead of sorting players into those who play “up” and “down”, sort

\(^{19}\)As before, we include atoms for buyers at 0 and sellers at 1. So, this does not imply that buyers have positive value for all $m$ units or that sellers want or are able to sell $m$ units.

\(^{20}\)There are less restrictive ways in which one might generalize NAG. For example, if each buyer’s first value is uniform $[3, 4]$, and their second value is uniform $[0, v_{i1}]$ then there are many buyer values in each range. An example in Section 5.1 of Swinkels (2001) suggests that this is not strong enough to guarantee efficiency.
them into those who make 0 up bids, 1 up bid, etc. This is notationally intensive but straightforward and hence omitted.

Finally, we must show that since \(|u|, |c|, |l|\) shrink quickly, inefficiency in the market disappears as \(1/n\). A proof of this is in the appendix. To see the issues involved, note that for the single unit case (and for the first unit of demand in the multiple unit case), a buyer’s impact on the price is small for two reasons. First, he is unlikely to be pivotal. Second, even if he is pivotal, he doesn’t affect the price much, since the next bid up is likely to be close. We exploit both of these forces in showing Lemma 6 and Lemma 8 and their adaptations here.

For units of demand after their first, many buyers can simultaneously be in the position that in raising bids other than their first, they pay more for units they were already winning. To get around this, consider the deviation to honest bidding. In any given realization, let \(x = u - c\). This is the maximum impact of \(i\) raising his \(m\) bids on price. If \(v_i < c\), then the deviation is irrelevant.

If \(c \leq v_i \leq c + 2mx\), then \(i\) may not benefit very much from any new unit won by raising \(b_i\), and may hurt himself by raising the price by as much as \(x\) on each of \(m - 1\) units already being won. But, critically, because of NAA, the number of \(v_i\) in \((c, c + 2mx)\) is only like \(nx\) (as always, ignoring constants). So, the expected cost to bidders from this case is like \(E(nx^2)\). But, the modified versions of Lemma 6 and Lemma 8 give that \(E(nx^2)\) is like \(1/n\).\(^{21}\) And, the expected efficiency loss from such players not winning also falls like \(1/n\).

Consider objects with \(v_i\) above \(c + 2mx\) where \(i\) is already winning an \(h\)th object. As before, for only one player can one of the associated bids be \(c\). So, the sum of costs in terms of raising these bids is at most \(x\). And, \(E(x) \leq 1/n\) as well.

The remaining objects have \(v_i\) above \(c + 2mx\) but are not winning. But, then the deviation to \(v\) wins an extra object at price at most \(c + x\), and raises the price by at most \(x\) on \(m - 1\) units, for a net profit of \(v_i - c - mx\). The efficiency loss from \(i\) not winning object \(h\) is at most \(v_i - c\), which, given that \(v_i - c > 2mx\), is at most twice \(v_i - c - x\). So, on these objects, bidder’s profits from the deviation are at least half of the efficiency loss on these units. Since costs from raising bids on other units are insignificant, it follows that the efficiency loss on these units is small since otherwise bidders will in aggregate have a profitable deviation. As the efficiency loss on other units...

\(^{21}\) The lower bound on the distribution function of \(x\) will be used to bound its expectation in our proof of Theorem 2.
units is also small, we are done.

6. Extensions

6.1. One-sided Uniform-price Auctions

Swinkels (2001) considers large one-sided auctions with independent values and a little bit of “noise.” An example is if there is a small independent probability that each player sleeps through the auction. In the uniform price case, it is shown that with the noise, the impact that any given player has on the price grows small in expectation. But then, since “honest” bidding has a small effect on the price paid, it must also have little benefit in winning extra objects. This implies asymptotic efficiency (without a rate of convergence).

An easy extension to the arguments here shows that a one-sided uniform price auction with $z$-independent values converges to efficiency at rate $1/n^{2-\alpha}$, even without noise. This paper thus significantly generalizes Swinkels (2001) for the uniform price case. The key is that here we think of “cost” as the impact on price in circumstances where the player affecting the price cares. This is a simpler object to bound, allowing both the greater generality, and fast convergence.22

6.2. Weaker Information Assumptions

We can weaken the information assumptions considerably and still obtain convergence. There is no problem if most players have considerably more knowledge about each other’s values than $z$-independence allows. What counts (for convergence) is that from the point of view of a non-vanishing fraction of players, there are “lots” of players who he cannot predict precisely, and that NAG applies to this set of players. It seems unlikely, however, that these weakenings would permit a result on rates of convergence.

6.3. Non-Private Values

We can also weaken the assumption of private values somewhat. Assume that an $\varepsilon$ fraction of the players have private values, and the remainder some sort of common. The arguments above show that over relevant ranges, the players with private values bid close to value. NAG implies that their

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22 The stronger notion of vanishing impact is needed to prove results for discriminatory auctions, which are also analyzed in that paper.
bids are then closely packed almost surely. Thus, the impact of bids on price disappears for all players. But then, common value types should bid nearly “honestly” (their bid should nearly equal the expected value of object conditional on being pivotal). Formalizing and generalizing such a result is left to future work.

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APPENDIX

Proofs for Section 3.2

Proof of Lemma 1: Wlog, let $K = \{1, 2, \ldots, |K|\}$. Let $P_K$ and $P_{N\setminus K}$ be the marginals of $P$ on $K$ and $N \setminus K$ respectively, and let $P_{K \times N \setminus K}$ be the associated product measure. Fix a rectangular event $F_K = F_1 \cap F_2 \cap \ldots \cap F_{|K|}$, where each $F_i$ only involves $v_i$. Then,

$$\Pr(F_K | F_{N\setminus K}) = \prod_{i=1}^{|K|} \Pr(F_i | F_{i+1} \cap \ldots \cap F_{|K|} \cap F_{N\setminus K})$$

$$\leq z^{-|K|} \prod_{i=1}^{|K|} \Pr(F_i | F_{i+1} \cap \ldots \cap F_{|K|}) \quad \text{(using } z - \text{independence})$$

$$= z^{-|K|} \Pr(F_K) = z^{-|K|} P_K(F_K).$$

Analogously,

$$\Pr(F_K | F_{N\setminus K}) \geq z^{|K|} P_K(F_K).$$

These inequalities extend to any $F_K$ in the product $\sigma$-algebra, as such a set is the limit of a countable union of rectangles. Thus

(A.1) $z^{-|K|} \Pr(F_K) \Pr(F_{N\setminus K}) \geq \Pr(F_K \cap F_{N\setminus K}) \geq z^{|K|} \Pr(F_K) \Pr(F_{N\setminus K})$

or equivalently

(A.2) $z^{-|K|} P_{K \times N \setminus K} \geq P \geq z^{|K|} P_{K \times N \setminus K}.$

Let $F_K$ and $F_{N\setminus K}$ be events about values and bids. Then

$$\Pr(F_K \cap F_{N\setminus K}) = \int_{[0,1]^{|N|}} \Pr(F_K | v_K) \Pr(F_{N\setminus K} | v_{N\setminus K}) dP$$

$$\leq z^{-|K|} \int_{[0,1]^{|N|}} \Pr(F_K | v_K) \Pr(F_{N\setminus K} | v_{N\setminus K}) dP_{K \times N \setminus K}$$

$$= z^{-|K|} \int_{[0,1]^{|K|}} \Pr(F_K | v_K) dP_K \int_{[0,1]^{|N|-|K|}} \Pr(F_{N\setminus K} | v_{N\setminus K}) dP_{N\setminus K}$$

$$= z^{-|K|} \Pr(F_K) \Pr(F_{N\setminus K}).$$

The product in the first integral is defined by the players’ distributional strategies. The second line uses (A.2). The third line applies Fubini’s Theorem. The final line integrates. Similarly $\Pr(F_K \cap F_{N\setminus K}) \geq z^{|K|} \Pr(F_K) \Pr(F_{N\setminus K})$ and so (A.1) holds for all events.
Similarly, for rectangular events $F_{N \setminus K}$,

\[(A.3) \quad z^{-|N \setminus K|} \Pr(F_K) \Pr(F_{N \setminus K}) \geq \Pr(F_K \cap F_{N \setminus K}) \geq z^{\frac{|N \setminus K|}{|K|}} \Pr(F_K) \Pr(F_{N \setminus K}). \]

Exchanging $K$ and $N \setminus K$ in the above and combining,

\[(A.4) \quad z^{-a} \Pr(F_K) \Pr(F_{N \setminus K}) \geq \Pr(F_K \cap F_{N \setminus K}) \geq z^a \Pr(F_K) \Pr(F_{N \setminus K}) \]

Dividing through by $\Pr(F_{N \setminus K})$ gives (2).

Let $X_K$ be a step function with values $x^a$ on a finite partition $\{F^a\}_{a \in A}$ where each $F^a$ is an event on bids and values in $K$. By the definition of conditional expectation $E(X_K | F_{N \setminus K}) = \sum_{a \in A} x^a \Pr(F^a | F_{N \setminus K})$. Thus by (2)

$$E(X_K | F_{N \setminus K}) \leq z^{-a} \sum_{a \in A} x^a \Pr(F^a) = z^{-a} E(X_K).$$

Analogously, $E(X_K | F_{N \setminus K}) \geq z^a E(X_K)$.

As an arbitrary $X_K$ is the limit of such step functions, (3) follows.

\[Q.E.D.\]

Proofs for Section 3.3

Proof of Lemma 2 : Wlog, let $K = \{1, 2, ..., \kappa\}$. Define the Bernoulli process with $\kappa$ independent trials with success probability $\overline{p}_i$ in trial $i$. Let $x_i \in \{0, 1\}$ be the outcome of trial $i$ and let $X^k = \sum_{i=1}^{k} x_i$. We claim that $X_K \equiv X^k$ FOSD $Q_K$ given $F_{N \setminus K}$. The proof is inductive. Let $Q^k$ be the number of $F_1,...,F_k$ that occur. Trivially, $X^0$ FOSD $Q^0$, since both are identically 0. Suppose $X^{k-1}$ FOSD $Q^{k-1}$ given $F_{N \setminus K}$. Then, for $r \in \{0, \ldots, k\}$,

$$\Pr(Q^k \leq r | F_{N \setminus K}) = \Pr(Q^{k-1} < r | F_{N \setminus K})$$

$$+ \Pr(F^k \{Q^{k-1} = r\} \cap F_{N \setminus K}) \Pr(Q^{k-1} = r | F_{N \setminus K})$$

$$\geq \Pr(Q^{k-1} < r | F_{N \setminus K})$$

$$+ (1 - \overline{p}_k) \Pr(Q^{k-1} = r | F_{N \setminus K})$$

$$= \overline{p}_k \Pr(X^{k-1} < r | F_{N \setminus K}) + (1 - \overline{p}_k) \Pr(X^{k-1} \leq r | F_{N \setminus K})$$

$$\geq \overline{p}_k \Pr(X^{k-1} < r) + (1 - \overline{p}_k) \Pr(X^{k-1} \leq r)$$

$$= \Pr(X^k \leq r).$$

The first inequality uses the definition of $\overline{p}_k$ and (4). The middle equality uses $\Pr(Q^{k-1} = r) = \Pr(Q^{k-1} \leq r) - \Pr(Q^{k-1} < r)$ and the final inequality uses the inductive hypothesis.
Similarly, if $Y_K$ is the number of successes in a Bernoulli process with success probabilities $p_i$, then given $F_{N \setminus K}, Q_K$ FOSD $Y_K$.

We want a large-deviations inequality for the bounding Bernoulli processes. As $X_K$ is a sum of non-identical independent Bernoulli trials, a slight alteration to the usual proof of Cramér’s Theorem (e.g., Shirayev (1996) p.68) is necessary. Let $\pi = \frac{1}{K}\sum_i p_i$. Then, for any $\phi \geq 0$ and $\lambda > 0$,
\[
\Pr \left( \frac{X_K}{\kappa \pi} > \phi \right) = \Pr \left( e^{\lambda X_K / \kappa \pi} \geq e^{\lambda \phi} \right) \leq \frac{E \left( e^{\lambda X_K / \kappa \pi} \right)}{e^{\lambda \phi}}
\]
by Markov’s inequality. Note also that $\Pr \left( \frac{X_K}{\kappa \pi} > \phi \right) = 0$ trivially when $\pi \phi \geq 1$. So, assume that $\pi \phi < 1$.

Now, as $X_K$ is a sum of independent random variables
\[
(A.5) \quad E e^{\lambda X_K / \kappa \pi} = \prod_{i \in K} \left( 1 - \overline{p}_i + \overline{p}_i e^{\lambda / \kappa \pi} \right)
\]
\[
= \exp \left( \log \prod_{i \in K} \left( 1 - \overline{p}_i + \overline{p}_i e^{\lambda / \kappa \pi} \right) \right)
\]
\[
= \exp \left( \sum_{i \in K} \log \left( 1 - \overline{p}_i + \overline{p}_i e^{\lambda / \kappa \pi} \right) \right)
\]
\[
\leq \exp \left( \kappa \log \left( 1 - \pi + \pi e^{\lambda / \kappa \pi} \right) \right)
\]
since $\log \left( 1 - x + xe^{\lambda / \kappa \pi} \right)$ is concave in $x$.

Thus,
\[
(A.6) \quad \Pr \left( \frac{X_K}{\kappa \pi} > \phi \right) \leq \exp \left[ -\lambda \phi + \kappa \log \left( 1 - \pi + \pi e^{\lambda / \kappa \pi} \right) \right]
\]
\[
= \exp \left[ -\kappa \left\{ \frac{\lambda}{\kappa \pi} \phi \pi - \log \left( 1 - \pi + \pi e^{\lambda / \kappa \pi} \right) \right\} \right]
\]
\[
= \exp \left[ -\kappa \left\{ \phi \pi (s \phi \pi - \log(1 - \pi + \pi e^s)) \right\} \right],
\]
where $s \equiv \frac{\lambda}{\kappa \pi}$. Given that $\lambda > 0$ was arbitrary, this holds for all $s > 0$, and so in particular for $s = \log \left( \frac{(1 - \pi)\phi}{1 - \pi \phi} \right)$ (this is positive, because $\pi \phi < 1$), yielding
\[
\Pr \left( \frac{X_K}{\kappa \pi} > \phi \right) \leq \exp \left[ -\kappa \left\{ \log \left( \frac{(1 - \pi)\phi}{1 - \pi \phi} \right) \phi \pi - \log \left( 1 - \pi + \pi e^{\log \left( \frac{(1 - \pi)\phi}{1 - \pi \phi} \right)} \right) \right\} \right]
\]
\[
= \exp \left[ -\kappa \left\{ \phi \pi \log \phi + (1 - \phi \pi) \log \left( \frac{1 - \phi \pi}{1 - \pi} \right) \right\} \right]
\]
As \( \log x \geq (x - 1)/x \) the second term in the braces is at least \( \pi(1 - \phi) \). Thus,

\[(A.7) \quad \Pr \left( \frac{X_K}{\kappa \pi} > \phi \right) \leq \exp \left[ -\kappa \pi (\phi \log \phi + 1 - \phi) \right].\]

Choosing \( \phi = 3 \),

\[(A.8) \quad \Pr \left( X_K > 3 \pi \right) \leq e^{-\kappa \pi(3 \log 3 - 2)} \leq e^{-\kappa \pi}.\]

Note that \( \frac{1}{z} E(Q_K) \geq \sum_{i \in K} \mathbb{P}_i = \kappa \pi \), and hence \( \Pr (X_K > 3 \pi) \geq \Pr (X_K > \frac{3}{z} E(Q_K)) \). And, \( \sum_{i \in K} \mathbb{P}_i \geq E(Q_K) \), and so \( e^{-\kappa \pi} \leq e^{-E(Q_K)} \). Finally, \( X_K \) stochastically dominates \( Q_K \). Taken together with (A.8), this implies

\[ \Pr \left( Q_K > \frac{3}{z} E(Q_K) \right) \leq e^{-E(Q_K)} \]

giving (6).

The proof for \( Y_K \) is similar: Define \( \pi = \sum_{i \in K} \mathbb{P}_i \). Then, for any \( \lambda < 0 \), and \( 0 < \phi < 1 \)

\[ \Pr \left( \frac{Y_K}{\kappa \pi} < \phi \right) = \Pr \left( e^{\lambda X_K/\kappa \pi} \geq e^{\lambda \phi} \right) \leq \frac{E \left( e^{\lambda X_K/\kappa \pi} \right)}{e^{\lambda \phi}}. \]

The derivation of (A.5) and (A.6) is then as before, replacing \( \mathbb{P}_i \) by \( \mathbb{P}_i \) and \( \Pr \left( \frac{X_K}{\kappa \pi} > \phi \right) \) by \( \Pr \left( \frac{Y_K}{\kappa \pi} < \phi \right) \). Note in particular that since \( \lambda < 0 \), \( s = \frac{\lambda}{\kappa \pi} \) can once again take on any positive value. Setting \( s = \log \left( \frac{1 - \pi}{1 - \pi \phi} \right) \)

is once again valid, as \( \phi < 1 \). Hence we arrive at the analog to (A.7):

\[(A.9) \quad \Pr \left( \frac{Y_K}{\kappa \pi} < \phi \right) \leq \exp \left[ -\kappa \pi (\phi \log \phi + 1 - \phi) \right].\]

Note that \( \sum_{i \in K} \mathbb{P}_i \geq z E(Q_K) \), so that \( \Pr (Y_K < \phi z E(Q_K)) \leq \Pr (Y_K < \phi \kappa \pi) \) and \( \exp \left[ -\kappa \pi (\phi \log \phi + 1 - \phi) \right] \leq \exp \left[ -z E(Q_K) (\phi \log \phi + 1 - \phi) \right] \). So,

\[(A.10) \quad \Pr (Q_K < \phi z E(Q_K)) \leq e^{-z E(Q_K) (\phi \log \phi + 1 - \phi)}. \]

Since \( \frac{1}{z} \log \frac{1}{z} + 1 - \frac{1}{z} > 0.3 \), (5) follows.

**Proof of Corollary 1:** Note that \( \Pr (Q_K = 0 | F_{N \setminus K}) \leq \Pr (Q_K \leq \phi E(Q_K)) \) for any \( \phi > 0 \). Equation A.10 then gives

\[ \Pr (Q_K = 0 | F_{N \setminus K}) \leq e^{-z E(Q_K) (\phi \log \phi + 1 - \phi)} \leq e^{-z \Pr (Q_K \geq 1) (\phi \log \phi + 1 - \phi)}. \]
since $E(Q_K) \geq \Pr(Q_K \geq 1)$. As this holds for $\phi$ arbitrarily close to 0 and as $\lim_{\phi \to 0} \phi \log \phi = 0$,
\[
\Pr \left( Q_K \geq 1 \mid F_N \right) \geq 1 - e^{-z \Pr(Q_K \geq 1)} = \frac{1 - e^{-z \Pr(Q_K \geq 1)}}{\Pr(Q_K \geq 1)} \Pr(Q_K \geq 1)
\]

For $x \in (0, 1]$, $(1 - e^{-zx})/x$ is minimized at $x = 1$. Q.E.D.

Proofs for Section 3.4

Proof of Lemma 3: Let $[x]$ denote the integer part of $x$. Partition $[0, 1]$ into $k \equiv \lfloor n^{1-\alpha/4} \rfloor$ intervals $\{I_\kappa\}$ of equal length (between $1/n^{1-\alpha/4}$ and $1/2n^{1-\alpha/4}$). Let $Q_B(I_\kappa)$ be the number of buyers with values in $I_\kappa$. Note that
\[
W_n/k = W_n \mid I_\kappa \mid \geq E(Q_B(I_\kappa)) \geq w_n \mid I_\kappa \mid = w_n/k.
\]

Let $E_{1\kappa} \equiv \{ \frac{3}{\pi} W_n/k \geq Q_B(I_\kappa) \geq \frac{3}{\pi} w_n/k \}$. By Lemma 2,
\[
\Pr(E_{1\kappa}) \geq 1 - 2e^{-3zw_n/k} \geq 1 - \frac{1}{n^5}
\]
for $n$ SL, since $n/k \to n^{\alpha/4}$. Similarly, let $Q_S(I_\kappa)$ be the number of sellers with values in $I_\kappa$, and define $E_{2\kappa} = \{ \frac{3}{\pi} W_n/k \geq Q_S(I) \geq \frac{3}{\pi} w_n/k \}$. Then,
\[
\Pr(E_{2\kappa}) \geq 1 - \frac{1}{n^5}
\]
for $n$ SL.

Let $\hat{N} \equiv \cap_{\kappa} (E_{1\kappa} \cap E_{2\kappa})$. Then, for $n$ SL,
\[
(A.11) \quad \Pr(\hat{N}) \geq 1 - \sum_{\kappa} \Pr(E_{1\kappa}) - \sum_{\kappa} \Pr(E_{2\kappa})
\]
\[
(A.12) \quad \geq 1 - 2 \frac{n^{1-\alpha/4}}{n^5} \geq 1 - 1/n^4.
\]
The first inequality holds as we are just double counting times where $\hat{N}$ does not hold.

Finally, note that for $n$ SL, any interval $I$ of length at least $\frac{1}{n^{1-\alpha/4}}$ contains at least $k \mid I \mid - 2 \geq k \mid I \mid / 2$ elements of $\{I_\kappa\}$. So, given $\hat{N}$,
\[
Q_B(I) \geq \frac{k \mid I \mid}{2} \frac{z}{3} w_n/k = \frac{nzw}{6} \mid I \mid.
\]

Similarly, $I$ intersects with at most $2k \mid I \mid$ elements of $\{I_\kappa\}$ and so $Q_B(I) \leq \frac{6}{\pi} W_n \mid I \mid$. The argument for $Q_S(I)$ is analogous. Thus, in fact $\hat{N} \subset \hat{N}$ and we are done.

Q.E.D.
Proofs for Section 4.2

To prove Proposition 1 we need a technical lemma.

Lemma 9: Let a subsequence of auctions \( \{A^n_t\}_{t=0}^\infty \) and associated equilibria be given. If \( \frac{E(V|T)}{nt} \geq \gamma \) for some \( \gamma > 0 \) and all \( t \), then \( \Pr(T) \to 1 \) along this subsequence.

Intuitively, if many players trade given \( T \), then many players must occasionally be bidding in a fairly aggressive way. But then, by \( z \)-independence, at least a fraction of them will be doing so almost all the time. The proof is more complicated because the event \( T \) is linked to all players’ actions, and so \( z \)-independence does not immediately apply.

Proof of Lemma 9: Choose a subsequence \( \{n_t\} \) such that \( E(V|T) \geq n_t \gamma \) for all \( t \). (We will suppress the subscript in what follows.) Let \( \gamma_0 = \gamma/2 \).

Then, since \( E(V|T) \geq \gamma \), \( \Pr(V > \gamma' n|T) > \gamma' \) since \( V \leq n \). Given the event \( \{V > \gamma' n\} \), if one selects \( \gamma' n_2 \) of the buyers at random, the probability that none trades is at most \( (1 - \gamma')(\gamma/2)^n_2 \leq 1/8 \) for \( n \) SL, and so there is a 7/8 probability that at least one such buyer is trading. Since this is true in expectation, it must be true for some particular set \( G_B \) of \( \gamma' n_2 \) buyers. Similarly, there is a set \( G_S \) of \( \gamma' n/2 \) sellers such that conditional on \( \{V > \gamma' n\} \) at least one seller in \( G_S \) is a trader with probability 7/8. Let \( G \equiv G_S \cup G_B \), and let \( T_G \subset T \) be the event that at least one buyer in \( G \) trades and one seller in \( G \) trades. Then, \( \Pr(T_G|\{V > \gamma' n\}) \geq 1 - 2(1/8) = 3/4 \), and so

\[
(A.13) \quad \Pr(T_G \cap \{V > \gamma' n\}) \geq 3/4 \Pr(V > \gamma' n).
\]

But then,

\[
\Pr(\{V > \gamma' n\} | T_G) = \frac{\Pr(T_G \cap \{V > \gamma' n\})}{\Pr(T_G)} \geq \frac{3/4 \Pr(V > \gamma' n)}{\Pr(T)} \geq 3\gamma'/4,
\]

where the first inequality uses (A.13) and \( T_G \subset T \), while the second uses \( \Pr(V > \gamma' n) > \gamma' \).

Let \( X \) be the event that at least \( \gamma' n_2 \) buyers and \( \gamma' n_2 \) sellers in \( N \setminus G \) trade. Since \( G \) has only \( \gamma' n_2 \) buyers or sellers, \( T_G \cap \{V > \gamma' n\} \) implies \( X \). So, \( \Pr(X|T_G) \geq 3\gamma'/4 \).
Let $p^*$ be such that $\Pr(p \geq p^* | X \cap T_G) \geq \frac{1}{2}$ and $\Pr(p \leq p^* | X \cap T_G) \geq \frac{1}{2}$. Let $Q_S$ be the number of sellers in $N \setminus G$ with $b_i \leq p^*$ and $Q_B$ the number of buyers in $N \setminus G$ with $b_i \geq p^*$. Then,

$$E(Q_S | T_G) \geq \Pr(X \cap \{p \leq p^*\} | T_G) \frac{\gamma'n}{2} \geq \frac{1}{2} \Pr(X | T_G) \frac{\gamma'n}{2} \geq 3\gamma^2 n/16,$$

and so

$$E(Q_S) = \sum_{i \in N_S \setminus G} \Pr(b_i \leq p^*) \geq z \sum_{i \in N_S \setminus G} \Pr(b_i \leq p^* | T_G) = zE(Q_S | T_G) = 3z\gamma^2 n/16.$$

Thus, by Lemma 2, $\Pr(Q_S = 0) \to 0$ as $n \to \infty$. Similarly $\Pr(Q_B = 0) \to 0$.

Thus, along this subsequence $\Pr(Q_S > 0 \text{ and } Q_B > 0) \to 1$. When $Q_S > 0$ and $Q_B > 0$ there is at least one buy bid above a sell bid so $\Pr(T) \to 1$.

Q.E.D.

PROOF OF PROPOSITION 1: Fix $A^g$ and a non-trivial equilibrium. Let $\phi_B \equiv \max_{N_B} b_i$ be the highest buy bid submitted and let $\phi_S \equiv \min_{N_S} b_i$ be the lowest sell bid. Note that $\Pr(\phi_B \geq x)$ is decreasing and continuous from the left. Similarly, $\Pr(\phi_S \leq x)$ is increasing and continuous from the right.

Let $v^* \in [0,1]$ have the property that for all $x \in [0,v^*)$, $\Pr(\phi_B \geq x) \geq \Pr(\phi_S \leq x)$, while for all $x \in (v^*, 1]$, $\Pr(\phi_B \geq x) \leq \Pr(\phi_S \leq x)$. Let

$$\delta \equiv \min \{ \Pr(\phi_B \geq v^*), \Pr(\phi_S \leq v^* \}.$$

Note that $\Pr(\phi_B > v^*) \leq \delta$. This is trivial if $\Pr(\phi_B > v^*) = \delta$. If $\Pr(\phi_B > v^*) > \delta$, then $\Pr(\phi_S \leq v^*) = \delta$. But then since $\Pr(\phi_S \leq x)$ is continuous from the right,

$$\Pr(\phi_B > v^*) = \lim_{v \downarrow v^*} \Pr(\phi_B \geq v) \leq \lim_{v \downarrow v^*} \Pr(\phi_S \leq v) = \Pr(\phi_S \leq v) = \delta.$$

Analogously, $\Pr(\phi_S < v^*) \leq \delta$.

Assume that $\Pr(\phi_S \leq v^*) = \delta$. Then, $\Pr(T \cap \{p \leq v^*\}) \leq \Pr(\phi_S \leq v^*) = \delta$, while $\Pr(T \cap \{p > v^*\}) \leq \Pr(\phi_B > v^*) \leq \delta$. Similarly, if $\Pr(\phi_B \geq v^*) = \delta$, then, $\Pr(T \cap \{p < v^*\}) \leq \Pr(\phi_S < v^*) \leq \delta$, while $\Pr(T \cap \{p \geq v^*\}) \leq \Pr(\phi_B \geq v^*) = \delta$. So $\Pr(T) \leq 2\delta$.

Now, $\{\phi_S \leq v^*\} = \cup_{i \in N_S} \{b_i \leq v^*\}$. Hence, by Corrolary 1,

$$(A.14)\quad \Pr(\phi_S \leq v^* | F_{N_B}) \geq (1 - e^{-2}) \Pr(\phi_S \leq v^*) \geq (1 - e^{-2}) \delta$$

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for any $F_{NB}$. So,

\begin{align}
\Pr(T) & \geq \Pr(\{\phi_B \geq v^*\} \cap \{\phi_S \leq v^*\}) \\
& \geq \Pr(\phi_S \leq v^* \mid \phi_B \geq v^*) \Pr(\phi_B \geq v^*) \\
& \geq (1 - e^{-\varepsilon})\delta^2.
\end{align}

Assume that $v^* \leq 1/2$. (If not, the proof below applies, mutatis mutandis, to the sellers). Fix an arbitrary buyer $i$. Let $\phi_B^i \equiv \max_{N_B \setminus \{i\}} b_i$. Now,

\[\Pr(\phi_B^i < 2/3) = 1 - \Pr(\phi_B^i \geq 2/3) \geq 1 - \Pr(\phi_B^i \geq v^*),\]

and earns at least $1/6$ when he does so. So,

\[B_i \geq \Pr(v_i \in J) z (1 - \delta) (1 - e^{-\varepsilon}) \delta \frac{1}{6} - \pi_i,\]

where $\pi_i$ is $i$’s expected equilibrium profit.

Summing across buyers, and applying Lemma 4,

\begin{align}
A.17 & \Pr(\phi_S \leq v^* \mid v_i \in J, \phi_B^i < 2/3) \geq (1 - e^{-\varepsilon})\delta.
\end{align}

Let $J \equiv [5/6, 1]$. By Lemma 1,

\begin{align}
A.16 & \Pr(\phi_B^i < 2/3 \mid v_i \in J) \geq z (1 - \delta).
\end{align}

By (A.14),

\begin{align}
A.17 & \Pr(\phi_S \leq v^* \mid v_i \in J, \phi_B^i < 2/3) \geq (1 - e^{-\varepsilon})\delta.
\end{align}

Let $d_i$ be the deviation for $i$ that whenever $v_i \in J$ and the original strategy specified a bid below $v^*$, he bids $2/3$ instead. Under this strategy, he wins an object with probability at least \( \Pr(\phi_B^i < 2/3, \phi_S \leq v^*, v_i \in J) \),

which by (A.16) and (A.17) is at least

\[\Pr(v_i \in J) z (1 - \delta) (1 - e^{-\varepsilon}) \delta,\]

and earns at least $1/6$ when he does so. So,

\[B_i \geq \Pr(v_i \in J) z (1 - \delta) (1 - e^{-\varepsilon}) \delta \frac{1}{6} - \pi_i,\]

where $\pi_i$ is $i$’s expected equilibrium profit.

Summing across buyers, and applying Lemma 4,

\begin{align}
A.18 & z (1 - \delta) (1 - e^{-\varepsilon}) \delta \frac{1}{6} \sum_{N_B} \Pr(v_i \in J) - \sum_{N_B} \pi_i \leq \Pr(T).
\end{align}

By Assumption 2, $\sum_{N_B} \Pr(v_i \in J) \geq \frac{1}{6} \omega n$ for $n \geq 6$. As the gains to a buyer from any given trade are at most 1, and $V$ buyers trade,

\[\sum_{N_B} \pi_i \leq \Pr(T) E(V \mid T).\]
Substituting into (A.18) gives

\[
z(1 - \delta)(1 - e^{-z})\delta \left(\frac{1}{6}\right)^2 wn - \Pr(T)E(V \mid T) \leq \Pr(T).
\]

Using \(\Pr(T) \leq 2\delta\), dividing through by \(2n\delta > 0\) and rearranging gives

\[
\frac{1}{72} wz(1 - \delta)(1 - e^{-z}) - \frac{1}{n} \leq \frac{E(V \mid T)}{n}.
\]

Suppose that the proposition were false. Then there exists a subsequence \(\{n_t\}\) such that \(\Pr(T) \to 0\) as \(t \to \infty\). Along this subsequence \(\delta \to 0\), because \(\Pr(T) \geq (1 - e^{-z})\delta^2\) (by (A.15)). Hence, along this subsequence the LHS above converges to \(wz \gamma(1 - e^{-z})/72 > 0\). By Lemma 9, therefore, \(\Pr(T) \to 1\) along this subsequence; a contradiction.

Q.E.D.

Proofs for Section 4.3

We will prove stronger results that will be useful when we turn to the multiple unit case. Fix an integer \(m \geq 1\). Redefine \(\overline{uv}\) as the \(m^{th}\) bid above \(\overline{v}\). As before, let \(ug \equiv (cg, \overline{ug})\). When \(m = 1\), we have the original case.

PROOF OF LEMMA 5: Let us show that for \(n\) SL, \(E(|ug|) \leq \frac{1}{n^{1-\alpha}}\). Assume this is false along a subsequence. Let \(L \equiv \mathcal{N} \cap \{|ug| > \frac{1}{2}E(|ug|)\}\). Then, since \(|ug| \leq 1,

\[
E(|ug|) \leq \Pr(\mathcal{N})E(|ug| \mid \mathcal{N}) + \frac{1}{n^4} \quad \text{(for n SL)}
\]

\[
= \Pr(L)E(|ug| \mid L) + \Pr(\mathcal{N} \setminus L)E(|ug| \mid \mathcal{N} \setminus L) + \frac{1}{n^4}
\]

\[
= \Pr(L)E(|ug| \mid L) + \frac{E(|ug|)}{2} + \frac{1}{n^4}.
\]

So, for \(n\) SL

\[
(A.19) \quad \Pr(L)E(|ug| \mid L) > \frac{E(|ug|)}{3}.
\]

Consider \(d_i(b_i, v_i) = \max\{b_i, v_i - \frac{E(|ug|)}{4}\}\), and assume \(L\) holds. Since \(|ug| > E(|ug|)/2\), any buyer in the top half of \(|ug|\) is a winner after \(d_i\), and at most \(m\) were winners before (since buyers bid at most \(v_i\), and by definition, there are only \(m\) bids in \([\overline{v}, \overline{ug}]\)). By Definition 2 (which applies since \(|ug|/2 > E(|ug|)/4 > \frac{1}{4n^{1-\alpha}} > \frac{1}{n^{1-\alpha}}\) for \(n\) SL), and given that \(L\) holds, the number
of new winners is at least \( w' \frac{1}{2} n |ug| - m \geq \frac{w'}{3} n |ug| \) for \( n \) SL. Each new winner earns at least \( \frac{1}{4} E(|ug|) \). So, using (A.19)

\[
\sum B_i \geq \frac{E(|ug|)}{4} \frac{w'}{8} n E(|ug| | L) \Pr(L) \\
\geq \frac{E(|ug|)}{4} \frac{w'}{8} n E(|ug|) = \frac{E(|ug|)^2}{96} w' n.
\]

But, by Lemma 4, \( \sum B_i \leq E(|ug|) \), and so

\[
\frac{E(|ug|)^2}{96} w' n \leq E(|ug|)
\]

or

\[
nE(|ug|) \leq \frac{96}{w'} w'.
\]

For \( n \) SL, this contradicts \( E(|ug|) \geq \frac{1}{n^{1-\alpha_4}} \). The argument for sellers is analogous.

**Proof of Lemma 6:** We proceed by contradiction. Note first that if \( x \leq \frac{1}{n^{1-\alpha_3/2}} \), then \( \frac{x^2 n^{2-\alpha_3}}{2} \geq 1 \), and the claim is vacuous. So, among some subsequence \( \{ n_t, x_t \} \), \( n_t \to \infty \), assume \( \Pr(|ug| > x_t) > \frac{1}{x_t n_t^{2-\alpha_3}} \) and \( x_t > \frac{1}{n_t^{1-\alpha_3/2}} \). Then, for \( n \) SL

\[
\Pr(\mathcal{N} \cap \{ |ug| > x \}) > \Pr(|ug| > x) - \frac{1}{n^4} > \Pr(|ug| > x)/2
\]

(omitting \( t \)). Consider \( d_i(b_i, v_i) = \max\{ b_i, v_i - x \} \). Now consider the event \( \mathcal{N} \cap \{ |ug| > x \} \). Under normality the number of buyers in the top half of \( ug \) is at least \( w'n x/2 \) (note that \( x/2 \geq \frac{1}{n^{1-\alpha_3/2}} > \frac{1}{n^{1-\alpha_3/2}} \) for \( n \) SL, so Definition 2 does apply). So, there are \( w'n x/2 - m > w'n x/3 \) new winners, each earning \( x/2 \). So,

\[
\sum_{\mathcal{N}S} B_i \geq \Pr(|ug| > x) \frac{w'n x^2}{6}.
\]

But, using Lemma 4 and Lemma 5, \( \sum B_i \leq \frac{1}{n^{1-\alpha_4}} \). So,

\[
\Pr(|ug| > x) \frac{w'n x^2}{6} \leq \frac{1}{n^{1-\alpha_4}},
\]

Rearranging

\[
\Pr(|ug| > x) \leq \frac{6}{w'n^{2-\alpha_4}}.
\]
For \( n \) SL, this contradicts \( \Pr(|ug| > x) > \frac{1}{x^{n^2-\alpha_3}} \). \( Q.E.D. \)

**Proof of Lemma 7:** For buyer \( i \), consider \( d_i(b_i, v_i) \equiv \max\{b_i, v_i - x\} \). If \( i \in L_B(x) \), then \( i \) wins an extra object and earns at least \( x \). By Lemma 4 and Lemma 5,

\[
x E(l_B(x)) \leq \sum B_i \leq E(|ug|) \leq \frac{1}{n^{1-\alpha_4}}
\]

and so

\[(A.20) \quad E(l_B(x)) \leq \frac{1}{xn^{1-\alpha_4}}.\]

establishing the first claim. Now, note that in each realization,

\[
Y_B(1/n) = \frac{1}{n} l_B(1/n) + \int_{1/n}^{1} l_B(x) dx.
\]

(The first term is the rectangle, and the second the “triangle” in a consumer surplus calculation for demand curve \( l_B(.) \) up to demand \( Q = l_B(1/n) \)). Therefore, by Fubini’s theorem

\[
E(Y_B(1/n)) = \frac{1}{n} E(l_B(1/n)) + \int_{1/n}^{1} E(l_B(x)) dx
\]

\[
\leq \frac{1}{n^{1-\alpha_4}} + \int_{1/n}^{1} \frac{1}{x n^{1-\alpha_4}} dx \quad \text{(using (A.20) twice)}
\]

\[
= \frac{1}{n^{1-\alpha_4}} (1 + \log n)
\]

\[
\leq \frac{1}{n^{1-\alpha_3}} \quad \text{(for } n \text{ SL)}
\]

which establishes the second claim. Repeat for sellers. \( Q.E.D. \)

**Proofs for Section 4.4**

**Proof of Lemma 8:** Suppose the lemma is false, so that there exists a sequence \( \{n_t\}, \{x_t\} \) satisfying \( n_t \to \infty \) and

\[(A.21) \quad x_t \geq 1/n_t^{1-\alpha_2/2} \]

such that \( \Pr(|cg| \geq x_t) \geq 1/(n_t^{2-\alpha_2} x_t^2) \) (the claim is vacuous for smaller \( x_t \)).

**Step 1: Sparse Intervals.**
Recall from the proof of Lemma 3 the partition of $[0, 1]$ into $k \equiv [n^{1-\alpha/4}]$ disjoint intervals $\{I_\kappa\}$ of equal length between $1/n^{1-\alpha/4}$ and $2/n^{1-\alpha/4}$. Let $M(I_\kappa)$ be the number of bids in $I_\kappa$. Let $\tilde{w} = zw'/24$. Say $I_\kappa$ is sparse if $E(M(I_\kappa)) < \tilde{w}n^{\alpha/4}$. Let $X$ be the set of sparse intervals. For $\kappa \in X$, let $E_3\kappa \equiv \{M(I_\kappa) \leq \frac{3}{2}\tilde{w}n^{\alpha/4}\}$ be the event that there are not “too many” bids in $I_\kappa$.

For any given $\tau \in [0, 1]$ consider the process in which at stage one, values and bids are drawn according to the distributional strategy $\mu$, and at stage two, each bid is randomly and independently replaced by a bid in $I_\kappa$ with probability $\tau$. Let $M_\tau(I_\kappa)$ be the random variable giving the number of bids in $I_\kappa$ for this process. Clearly, $M_\tau(I_\kappa)$ stochastically dominates $M(I_\kappa)$ for any $\tau$. Choose $\tau^*$ such that $E(M_{\tau^*}(I_\kappa)) = \tilde{w}n^{\alpha/4}$. Then, Lemma 2 implies that

$$\Pr(E_3\kappa) \geq \Pr\left(M_{\tau^*}(I_\kappa) \leq \frac{3}{2}\tilde{w}n^{\alpha/4}\right) \geq 1 - e^{-\tilde{w}n^{\alpha/4}} \geq 1 - \frac{1}{n^5}$$

for $n$ SL.

For $\kappa \notin X$, let $E_3\kappa \equiv \{M(I_\kappa) \geq \frac{3}{2}\tilde{w}n^{\alpha/4}\}$ be the event that there are not “too few” bids in $I_\kappa$. Lemma 2 implies that for $n$ SL,

$$\Pr(E_3\kappa) \geq 1 - e^{-3\tilde{w}n^{\alpha/4}} \geq 1 - \frac{1}{n^5}.$$

Let $\mathcal{N}^\prime \equiv \mathcal{N} \cap (\cap \kappa E_3\kappa)$. Arguing as in the proof of Lemma 3, for $n$ SL

$$\Pr(\mathcal{N}^\prime) \geq 1 - 3\frac{n^{1-\alpha/4}}{n^{5}} \geq 1 - 1/n^4.$$

**Step 2: Sparse Regions and the Endpoints of Competitive Gaps.** Assemble maximal groups of adjacent sparse intervals into sparse regions. Let $\{J^\lambda\}_{\lambda \in \Lambda}$ be the set of sparse regions that are longer than $\frac{2}{\tilde{w}}$. Note that $\Lambda$ is possibly empty, and that $|\Lambda| \leq n$. For $n$ SL, $\tilde{w}n^{\alpha/4} > 1$. So, given $\mathcal{N}^\prime$, for all $n$ SL, each non-sparse interval contains at least 1 bid and so $cg$ cannot contain a non-sparse interval; $cg$ can include at most a $J^\lambda$ and parts of the two non-sparse intervals immediately adjacent. These two intervals, having length at most $2/n^{1-\alpha/4}$ become arbitrarily short compared to $x \geq 1/n^{1-\alpha/2}$ (from
(8), \(\alpha_2 > 2\alpha/3\), and hence \(\alpha_2/2 > \alpha/4\). Hence, given \(N\), and for \(n\) SL, a competitive gap of length \(x\) must (a) have intersection of length at least \(x/2\) with some \(J^\lambda\), and (b) intersect at most one \(J^\lambda\).

Let \(J^\lambda_{y}, y \in [0,1]\) be the point \(y^{th}\) of the way up the interval \(J^\lambda\). We begin by showing that it is very unlikely that \(cg\) ends a long way from the end of a \(J^\lambda\).

**Claim 1:** For all \(n\) SL

\[
\Pr\left( cg \in \bigcup \lambda [J^\lambda_0, J^\lambda_{4/5}] \right) \leq \frac{1}{12n^2 - \alpha_2 x^2},
\]

\[
\Pr\left( cg \in \bigcup \lambda [J^\lambda_{4/5}, J^\lambda_1] \right) \leq \frac{1}{12n^2 - \alpha_2 x^2}.
\]

**Proof:** Suppose this fails and along a subsequence \(\Pr(H) > 1/(12n^2 - \alpha_2 x^2)\), where \(H = \{cg \in \bigcup \lambda [J^\lambda_0, J^\lambda_{4/5}]\} \cap N\). Consider the event \(\{cg \in [J^\lambda_0, J^\lambda_{4/5}]\} \cap N\) for some \(\lambda \in \Lambda\). Let \(y \equiv J^\lambda_1 - cg\). As \(cg \in [J^\lambda_0, J^\lambda_{4/5}]\), \(y/2 \geq x/20 \geq 1/(20n^{1-\alpha_2/2})\), by (A.21). Recall from (8) that \(\alpha_2 > (2/3)\alpha\), thus for \(n\) SL \(y/2 \geq 1/n^{1-\alpha/3}\). By Definition 2, therefore, the number of values in \([J^\lambda_1 - y/2, J^\lambda_1]\) is at least \(w'y/n/2\).

On the other hand, by Step 1, given \(N\), each \(I_k \subseteq [J^\lambda_1 - y, J^\lambda_1]\) includes at most \(\frac{3}{5} \cdot n^{\alpha/4} = \frac{3}{5} \cdot (zw')/24 \cdot n^{\alpha/4} = w'n^{\alpha/4}/8\) bids. For \(n\) SL, this implies that the number of bids in \([J^\lambda_1 - y, J^\lambda_1]\) is at most \(w'y/n/4\) (by the same argument as in the proof of Lemma 3). Thus, given \(\{cg \in [J^\lambda_0, J^\lambda_{4/5}]\} \cap N\), there are at least \(w'y/n/2 = w'y/n/4\) players with value in \([J^\lambda_1 - y/2, J^\lambda_1]\) but bid below \(cg = J^\lambda_1 - y\). But then,

\[Y_B(y/2) \geq \frac{w'y/n}{4}.\]

As \(cg \in [J^\lambda_0, J^\lambda_{4/5}]\), \(y \geq x/10\). So, for any given \(\lambda\), whenever \(\{cg \in [J^\lambda_0, J^\lambda_{4/5}]\} \cap N\),

\[Y_B(x/20) \geq Y_B(y/2) \geq \frac{w'n x^2}{800}.\]

Now, for \(n\) SL,

\[\Pr(H \cap N) \geq \Pr(H) - \frac{1}{n^4} \geq \frac{1}{2} \Pr(H),\]

since for \(n\) SL, \(\frac{1}{n^4} < \frac{1}{2} \cdot \frac{1}{12n^2 - \alpha_2 x^2} \leq \frac{1}{2} \Pr(H)\). Thus,

\[E\left(Y_B\left(\frac{x}{20}\right)\right) \geq \frac{w'n x^2}{1600 \cdot \Pr(H)}.
\]
By (A.21), for $n \ SL \ \frac{1}{20} > \frac{1}{n}$, and hence $Y_B(\frac{1}{n}) < Y_B(\frac{1}{n})$. However, $E(Y_B(1/n)) \leq \frac{1}{n^{1-\alpha_3}}$ by Lemma 7. Thus,

$$\frac{w'nx^2}{1600} Pr(H) \leq \frac{1}{n^{1-\alpha_3}}.$$  

Rearranging,

$$Pr(H) \leq \frac{1600}{n^2-\alpha_3x^2w'} \leq \frac{1}{12n^2-\alpha_3x^2}$$

for $n \ SL$.\textsuperscript{23} This contradicts our initial assertion. Repeat for sellers in the lower fifth to get the second claim. Q.E.D.

**Step 4:** Relative Probabilities of competitive and supporting gaps. Let $cg_\lambda \equiv \left\{ cg \supseteq \left[ J_{1/5}^\lambda, J_{4/5}^\lambda \right] \right\}$, and let $c_\lambda \equiv Pr(cg_\lambda)$. Let $lg_\lambda \equiv \left\{ lg \supseteq \left[ J_{1/5}^\lambda, J_{2/5}^\lambda \right] \right\}$, and $l_\lambda \equiv Pr(lg_\lambda)$. Finally, let $ug_\lambda \equiv \left\{ ug \supseteq \left[ J_{3/5}^\lambda, J_{4/5}^\lambda \right] \right\}$, and $u_\lambda \equiv Pr(ug_\lambda)$. We next show that for some $\lambda$, $c_\lambda$ is both non-trivial, and much larger than either $l_\lambda$ or $u_\lambda$.

**Claim 2:** For $n \ SL$, there exists $\lambda$ such that

(A.23) \[ c_\lambda > \frac{1}{n^4}, \]

and such that

(A.24) \[ \frac{l_\lambda + u_\lambda}{c_\lambda} \leq \frac{1}{n^{2-\alpha_3}/2}. \]

**Proof:** As $Pr(|cg| \geq x) > n^{\alpha_2-2x^{-2}}$ and $Pr(N') \geq 1-1/n^4$, for $n \ SL$ $Pr(\{|cg| \geq x \} \cap N') \geq \frac{2}{6}n^{\alpha_2-2x^{-2}}$. By Claim 1, the probability of a competitive gap in $J^\lambda$ not including the middle $3/5$ is also less than $\frac{1}{6}n^{\alpha_2-2x^{-2}}$ for $n \ SL$. Therefore, $\sum_{\lambda \in \Lambda} c_\lambda \geq \frac{1}{6}n^{\alpha_2-2x^{-2}}$. Let $\Lambda'$ denote the subset of regions with $c_\lambda > 1/n^4$. There are at most $n$ regions. Thus,

$$\sum_{\lambda \in \Lambda \setminus \Lambda'} c_\lambda \leq \frac{n}{n^4} \leq \frac{1}{6}n^{\alpha_2-2x^{-2}}$$

\textsuperscript{23}We remark again that we have chosen transparency over any attempt to keep the various constants in these arguments small.
for \( n \) SL and so
\[
\sum_{\lambda \in \Lambda'} c_{\lambda} \geq \frac{1}{2} n^{\alpha_2 - 2} x^{-2}.
\]

From Lemma 6, for \( n \) SL
\[
\sum_{\lambda \in \Lambda'} u_\lambda - \frac{1}{n^3} \leq \Pr \left( |u| > \frac{x}{5} \right) \leq 25 n^{\alpha_3 - 2} x^{-2}.
\]

The first inequality holds because under \( \mathcal{N}' \) every non-sparse interval contains at least one bid, and so \( u_{\lambda} \) and \( u_{\lambda'} \) are disjoint events. When \( \mathcal{N}' \) does not hold, the overcounting is at most \( |\Lambda'| \leq n \). As \( \Pr(\mathcal{N}'^c) \leq \frac{1}{n^4} \) for \( n \) SL, the result follows. The second inequality applies Lemma 7. But then,
\[
\sum_{\lambda \in \Lambda'} l_\lambda + u_\lambda \leq 50 n^{\alpha_3 - 2} x^{-2} + \frac{2}{n^3}
\]

Thus
\[
\frac{\sum_{\lambda \in \Lambda'} l_\lambda + u_\lambda}{\sum_{\lambda \in \Lambda'} c_{\lambda}} \leq \frac{50 n^{\alpha_3 - 2} x^{-2} + \frac{2}{n^3}}{\frac{1}{2} n^{\alpha_2 - 2} x^{-2}} \leq \frac{1}{n^{(\alpha_2 - \alpha_3)/2}}.
\]

for \( n \) SL. Since this is true on average, it must be true for at least one \( \lambda \in \Lambda' \).

Q.E.D.

In what follows, we refer to a \( \lambda \) for which Claim 2 holds. Let \( \tilde{c} \equiv \Pr(\sigma g \supseteq [J_{2/5}^\lambda, J_{3/5}^\lambda]) \) be the probability of a competitive gap including the middle fifth of \( J_\lambda \). We will show that \( \tilde{c} \) is close to 1. The idea is that the only way to have \( c_\lambda \) be large relative to \( u = u_\lambda \) and \( l = l_\lambda \) will be for players to behave essentially deterministically.

Let \( U_i \equiv \{ b_i \geq J_{2/5}^\lambda \} \) be the event that \( i \) bids up and \( \tilde{U}_i \equiv \{ b_i > J_{2/5}^\lambda \} \) be the event that \( i \) bids weakly up. Symmetrically, let \( D_i \equiv \{ b_i \leq J_{1/5}^\lambda \} \) and \( \tilde{D}_i \equiv \{ b_i \leq J_{3/5}^\lambda \} \) be the events that \( i \) bids down and weakly down. Let \( p_i \equiv \Pr(U_i), \tilde{p}_i \equiv \Pr(\tilde{U}_i), q_i \equiv \Pr(D_i) \) and \( \tilde{q}_i \equiv \Pr(\tilde{D}_i) \). Order the players so that \( \tilde{q}_1 \leq \tilde{q}_2 \leq ... \leq \tilde{q}_{2n} \).

**Step 5: A preliminary inequality.** Define
\[
A_{i-j} \equiv \cap_{j' > i, j' \neq j} D_{j'}
\]
as the event that all players after \( i \) not including \( j \) bid down. Then, for any event \( F \) involving 1, 2, ..., \( i \),

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Step 6: Two Bounds

Recall that \( c g \equiv \{ c g \supseteq [J_{1/5}^\lambda, J_{4/5}^\lambda] \} \). Let \( c g_{ij} \), \( i < j \), be the event \( c g \) where \( i \) and \( j \) are the two highest indexed players for whom \( U_i \) holds. Let \( c g_{ij} \). Let \( F^i \) be the event that \( U_{j'} \) holds for \( j' = i \) and for \( n - 2 \) other \( j' \in \{1, ..., i\} \), while \( D_{j'} \) holds for all other \( j' \in \{1, ..., i\} \).

Then,

\[
c_{ij}^\lambda = \Pr(F^i \cap A_{ij}^- \cap U_j) = \Pr(U_j | F^i \cap A_{ij}^-) \Pr(F^i \cap A_{ij}^-) = \frac{\Pr(D_j | F^i \cap A_{ij}^-) \Pr(U_j | F^i \cap A_{ij}^-) \Pr(F^i \cap A_{ij}^-)}{\Pr(D_j | F^i \cap A_{ij}^-) \Pr(U_j | F^i \cap A_{ij}^-) \Pr(F^i \cap A_{ij}^-)} \leq \frac{p_j}{z^2 d_j} \Pr(D_j \cap F^i \cap A_{ij}^-) \Pr(F^i \cap A_{ij}^-) = \frac{p_j}{z^2 d_j} \Pr(D_j \cap F^i \cap A_{ij}^-). \]

Recall that \( u g_\lambda \) is the event \( \{ u g \supseteq [J_{3/5}^\lambda, J_{4/5}^\lambda] \} \). Let \( u g_{ij}^i \) be the event \( u g_\lambda \) where \( i \) is the last player to bid up, and let \( u g_{ij}^i = \Pr(u g_{ij}^i) \). When \( D_j \cap F^i \cap A_{ij}^- \) holds, \( i \) is the last player to bid up and in total \( n - 1 \) players bid up while the rest bid weakly down. Thus, \( \{ D_j \cap F^i \cap A_{ij}^- \} \subseteq u g_{ij}^i \), and so
\[ \Pr(\tilde{D}_j \cap F_i \cap A_{-j}^i) \leq u^i_\lambda. \] The previous equation thus implies

\[ c^{ij}_\lambda \leq \frac{p_j}{z^j q_j} u^i_\lambda. \] (A.26)

Another bound on \( c^{ij}_\lambda \) comes from (A.25):

\[ c^{ij}_\lambda \leq \Pr(A_{-j}^i) \leq e^{-z(-1+\sum_{j'>i} p_{j'}). \] (A.27)

**Step 7: Up and Down Players.** We next show that for all \( n \) SL, \( \tilde{q}_i = \Pr(\tilde{D}_i) \leq \frac{1}{4} \) for \( i \leq n \) (these are the up players), and \( \tilde{p}_i = \Pr(\tilde{U}_i) \leq \frac{1}{4} \) for \( i > n \) (these are the down players).

Consider the first claim. Suppose that \( \tilde{q}_n > 0 \) (if \( \tilde{q}_n = 0 \) the result is immediate). Let \( c^i_\lambda \equiv \sum_{j>i} c^{ij}_\lambda \) be the probability that \( c g_\lambda \) occurs, where \( i \) is the second last up player. Let \( i^* \leq 2n \) be the last index with the property that \( \sum_{j'>i} p_{j'} > n(\alpha_2 - \alpha_3)/4 \). Then,

\[
\sum_{i \leq i^*} c^i_\lambda = \sum_{i \leq i^*} \sum_{j>i} c^{ij}_\lambda \\
\leq n^2 e^{-z(-1+\sum_{j'>i} p_{j'})} \quad \text{(using (A.27))}
\leq n^2 e^{-z(-1+n(\alpha_2 - \alpha_3)/4)}
\leq \frac{1}{2n^4} \quad \text{(for } n \text{ SL)}
\leq \frac{1}{2} c_\lambda \quad \text{(using (A.23))}
\]
So, as \( c_\lambda = \sum_{i \geq n-1} c^i_\lambda \), for \( n \) SL,
\[
\frac{1}{2} c_\lambda \leq \sum_{i > i^*, i \geq n-1} c^i_\lambda \\
\leq \sum_{i > i^*, i \geq n-1} \sum_{j > i} \frac{p_{ij}}{z^2 \hat{d}_j} u^i_\lambda \text{ (using (A.26))} \\
\leq \frac{1}{z^2 \hat{q}_n} \sum_{i > i^*, i \geq n-1} \sum_{j > i} p_j u^i_\lambda \text{ (since } \hat{q}_i \text{ is increasing)} \\
\leq \frac{n(\alpha_2 - \alpha_3)^4}{z^2 \hat{q}_n} \sum_{i > i^*, i \geq n-1} u^i_\lambda \text{ (by choice of } i^*) \\
\leq \frac{n(\alpha_2 - \alpha_3)^4}{z^2 \hat{q}_n} u_\lambda \\
\leq \frac{n(\alpha_2 - \alpha_3)^4}{z^2 \hat{q}_n} \frac{1}{n(\alpha_2 - \alpha_3)^2} c_\lambda \text{ (by (A.24))} \\
= \frac{1}{z^2 \hat{q}_n} \frac{1}{n(\alpha_2 - \alpha_3)^2} c_\lambda. \]

Comparing the first and last expressions, \( \hat{q}_n \to 0 \), and so in particular, \( q_i \leq 1/4 \) all \( i \leq n \) for all \( n \) SL.

If the players are ordered so that \( \hat{p}_i \) increases, this argument can be repeated considering events in which \( n-1 \) of the first \( i \) players bid down and the others bid up. Thus there are \( n \) players for whom \( \hat{p}_i \to 0 \). As \( \hat{p}_i + \hat{q}_i \geq 1 \) these players are disjoint from players \( 1, \ldots, n \), and so must be the players \( \{n + 1, \ldots, 2n\} \).

Let \( R \equiv \cap_{i \leq n} U_i \cap_{i > n} D_i = c g^{n-1}_\lambda \) be the event that all the players bid according to their type (and a competitive gap occurs).

**Step 8: A lower bound for \( \Pr(R) \):** We already know that \( \sum_{i \leq i^*, j > i} c^{ij}_\lambda \leq \frac{1}{2} c_\lambda \) for \( n \) SL. We will show that for \( n \) SL \( \sum_{i > i^*, j > n+1} c^{ij}_\lambda \leq \frac{1}{4} c_\lambda \). Since \( R \) is the only event left involving \( c g^{n-1}_\lambda \), it would then follow that \( \Pr(R) \geq \frac{c_\lambda}{4} \). So, as
in Step 7 note that

\[ \sum_{i \geq i^*, j \geq n+1} c_{\lambda}^{ij} \leq \sum_{i \geq i^*, j \geq n+1} \frac{p_j}{z^2q_j} u_{\lambda}^{ij} \]

\[ \leq \frac{1}{z^2q_{n+1}} \sum_{i > i^*} u_{\lambda} \sum_{j > i} p_j \text{ (note the } n+1) \]

\[ \leq \frac{n(\alpha_2 - \alpha_3)/4}{2z^2} \sum_{i > i^*} u_{\lambda} \text{ (since } q_{n+1} \geq 1 - \tilde{p}_{n+1} \geq \frac{3}{4}) \]

\[ \leq \frac{n(\alpha_2 - \alpha_3)/4}{2z^2} u_{\lambda} \]

\[ \leq \frac{n(\alpha_2 - \alpha_3)/4}{2z^2} \frac{1}{n(\alpha_2 - \alpha_3)/4 c_{\lambda}} \]

\[ = \frac{1}{2z^2 n(\alpha_2 - \alpha_3)/4 c_{\lambda}} \]

\[ \leq \frac{1}{4 c_{\lambda}} \text{ (for } n \text{ SL)}. \]

Thus, \( \Pr(R) \geq c_{\lambda}/4 \).

**Step 9: A Persistent Competitive Gap.** Let \( \tilde{R} \supseteq R \) be the event that all the players get it nearly right — the first \( n \) players are not bidding below \( J_{3/5}^{\lambda} \) and the others are not bidding above \( J_{2/5}^{\lambda} \). For \( i > n \), define \( R_{-i} \) to be the event that all players except \( i \) play according to type. If \( R_{-i} \) occurs and player \( i \) bids weakly up, then \( l_g \geq [J_{1/5}^{\lambda}, J_{2/5}^{\lambda}] \). Thus,

\[ u_{\lambda} \geq \sum_{i > n} \Pr(R_{-i}) \Pr(\tilde{U}_i|R_{-i}) \]

\[ \geq z \Pr(R) \sum_{i > n} \tilde{p}_i \text{ (by } z\text{-independence and since } R \subseteq R_{-i}) \]

\[ \geq \frac{zc_{\lambda}}{4} \sum_{i > n} \tilde{p}_i \text{ (by Step 8)}. \]

Since \( \frac{u_{\lambda}}{c_{\lambda}} \to 0 \),

\[ \sum_{i > n} \tilde{p}_i \to 0. \]

Arguing symmetrically,

\[ \sum_{i \leq n} \tilde{q}_i \to 0. \]
Thus, \[
\Pr(\tilde{R}) \geq 1 - \sum_{i \leq n} \tilde{q}_i - \sum_{i > n} \tilde{p}_i \to 1.
\]

Step 10: A Contradiction. When \(\tilde{R} \cap \mathcal{N}'\) occurs, \([J_{2/5}^\lambda, J_{3/5}^\lambda] \subseteq \mathcal{cg}\). And since the probability of trade is bounded away from 0, and since \(\Pr(\tilde{R}) \to 1\), there is at least one buyer in \(\{1, \ldots, n\}\) and at least one seller in \(\{n + 1, \ldots, 2n\}\).

Let \(p^*\) be the expected price conditional on \(\tilde{R}\). Either \(p^* \leq J_{1/2}^\lambda\), or \(p^* \geq J_{1/2}^\lambda\). Wlog, assume \(p^* \geq J_{1/2}^\lambda\). Let \(x^\lambda\) be the length of \(J^\lambda\). By construction, \(x^\lambda \geq \frac{1}{2}x \geq \frac{1}{2n^{1-\alpha_2}}\).

Assume first that \(J_1^\lambda \geq 1 - 3x^\lambda\). Consider any buyer in \(\{1, \ldots, n\}\). A bid of \(J_{2/5}^\lambda\) wins whenever \(\tilde{R}\) occurs, and forces the price to at most \(J_{2/5}^\lambda\). So, conditional on \(\tilde{R}\), the buyer’s expected gain from lowering the price is at least \(J_{1/2}^\lambda - J_{2/5}^\lambda \geq \frac{x^\lambda}{6}\). On the other hand, when \(\tilde{R}\) does not occur, he may go from being a winner to a loser. But, for this to happen, it must be that \(\mathcal{cg} \geq J_{2/5}^\lambda\). Then \(i\)’s lost profit is at most \(1 - \mathcal{cg} \leq 4x^\lambda\). Since \(\Pr(\tilde{R}) \to 1\),

\[
\Pr(\tilde{R}) \frac{x^\lambda}{6} - (1 - \Pr(\tilde{R}))4x^\lambda
\]

is eventually positive, and we have a contradiction.

Assume \(J_1^\lambda < 1 - 3x^\lambda\). Given \(\mathcal{N}'\), the number of buyers with value in \((J_1^\lambda + 2x^\lambda, J_1^\lambda + 3x^\lambda)\) is at least \(w'nx^\lambda\). But, by Lemma 7 for \(n\ SL\)

\[
E(\#U(x^\lambda)) \leq \frac{1}{n^{1-\alpha_4}x^\lambda}.
\]

Since \(x^\lambda \geq \frac{1}{2n^{1-\alpha_2}}\),

\[
\frac{w'nx^\lambda}{1/n^{1-\alpha_4}x^\lambda} \geq w'n^{\alpha_2-\alpha_4}.
\]

It follows that for \(n\ SL\), at least half the buyers with a value in \((J_1^\lambda + 2x^\lambda, J_1^\lambda + 3x^\lambda)\) trade conditional on \(\tilde{R} \cap \mathcal{N}'\) (and so bid above \(J_{3/5}^\lambda\)). Consider the deviation that any buyer with a value in \((J_1^\lambda + 2x^\lambda, J_1^\lambda + 3x^\lambda)\) and a bid above \(J^\lambda\) bids \(J_{2/5}^\lambda\) instead. Given \(\tilde{R} \cap \mathcal{N}'\), this gains the buyer at least \(x^\lambda/6\). Given \(\mathcal{N}'\), the number of players in \((J_1^\lambda + 2x^\lambda, J_1^\lambda + 3x^\lambda)\) is at least \(w'nx^\lambda\) and at most \(W'nx^\lambda\). So, given \(R \cap \mathcal{N}'\), the expected sum of gains is at least \(\frac{w'nx^\lambda x^\lambda}{6}\). The loss from such a buyer going from being a winner to a loser is again at most \(4x^\lambda\). Given \(\mathcal{N}' \setminus \tilde{R}\), there are at most \(W'nx^\lambda\) such buyers. In \(\mathcal{N}'c\), the worst
case is that all \( n \) buyers are in \((J_1^\lambda + 2x^\lambda, J_1^\lambda + 3x^\lambda)\). So, the expected sum of losses is at most

\[
\Pr(N \setminus \tilde{R}) W' n x^\lambda 4x^\lambda + \Pr(N^{\text{ce}}) n 4x^\lambda \leq \left(1 - \Pr(\tilde{R})\right) W' n 4(x^\lambda)^2 + \frac{1}{n^4} n 4x^\lambda
\]

and thus, since the deviation cannot be profitable,

\[
\Pr(\tilde{R}) \frac{w' n x^\lambda x^\lambda}{2} \leq \left(1 - \Pr(\tilde{R})\right) W' n 4(x^\lambda)^2 + \frac{1}{n^4} 4x^\lambda.
\]

Dividing both sides by \( n(x^\lambda)^2 \),

\[
\Pr(\tilde{R}) \frac{w' n x^\lambda x^\lambda}{2} \leq 4W' \left(1 - \Pr(\tilde{R})\right) + \frac{4}{n^2 x^\lambda}
\]

\[
\leq 4W' \left(1 - \Pr(\tilde{R})\right) + \frac{8}{n^2 x} \quad \text{(since } x^\lambda \geq \frac{1}{2} x)\]

\[
\leq 4W' \left(1 - \Pr(\tilde{R})\right) + \frac{8}{n^{1-\alpha/2}}
\]

\[
= 4W' \left(1 - \Pr(\tilde{R})\right) + \frac{8}{n^{1+\alpha/2}}.
\]

Since \( \Pr(\tilde{R}) \to 1 \), the LHS goes to \( w'/12 \), while the RHS goes to 0, a contradiction. Q.E.D.

**Proofs for Section 5**

Let \( x \) be the random variable \( \overline{w}_m - cg \). In an \( m \) unit demand/supply setting, this is the maximum impact of raising a buyer’s bid vector on price. Let \( p \) be the price. We will show that in expectation buyers achieve within \( \frac{1}{2n^{1-\alpha}} \) of the consumer surplus if they can price take at \( p \). A symmetric argument applies to sellers. But, the sum of consumer and producer surplus at an arbitrary \( p \) is at least as large as the surplus at the Walrasian price. So, this both establishes that the market achieves within \( 1/n^{1-\alpha} \) of the efficient surplus and that price must be asymptotically Walrasian, else the market achieves more than the feasible surplus, a contradiction. Finally, from NAG and NAA, expected feasible surplus grows like \( n \), and the result follows.

Consider, for buyers, the truth-telling deviation \( d_i(b_i, v_i) = v_i \), remembering that \( v_i \) and \( b_i \) are now vectors in \([0, 1]^m\). Let \( W \) be the set of \( ih \), \( i \in N_B \) that are allocated an object. Let \( Y_{ih} = 0 \) if \( i \) wins an object \( h \),

\[ ^{24} \text{Formally, the difference between the realized price and the competitive price must converge to 0 in probability.} \]
and let $Y_{ih} = \max[v_{ih} - p, 0]$ otherwise. So, $Y_{ih}$ gives the loss in consumer surplus compared to price taking at $p$ from $i$ not winning object $h$.

In any given realization, think about moving from $b_i$ to $v_i$ one bid at a time, starting from $b_{i1}$. Let $\hat{C}_{ih}$ be the cost to $i$ from raising bid $h$ in terms of raising the price paid on units already won, and $\hat{B}_{ih}$ the profit to $i$ of winning an extra unit.

If $v_{ih} < cg$, then raising $b_{ih}$ to $v_{ih}$ is irrelevant to both $p$ and the allocation. Hence, $\hat{B}_{ih} - \hat{C}_{ih} = 0$. And, since $v_{ih} < p$, $Y_{ih} = 0$.

If $v_{ih} \in [cg, \overline{mg} + 2mx]$, then raising $b_{ih}$ to $v_{ih}$ may raise the price on units already won by as much as $x$. So, $\hat{B}_{ih} - \hat{C}_{ih} \geq -(m - 1)x$. And, since $v_{ih} \in [cg, \overline{mg} + 2mx]$ and $p \geq cg$, $Y_{ih} \leq (2m + 1)x$. In any normal realization, the number of such $ih$ is at most $Knx$ for some $K < \infty$. In a non-normal realization, there are at most $nm$ values in this range. Hence, the expected number of such values is at most

$$\left(1 - \frac{1}{n^4}\right)Knx + \frac{1}{n^4}nm \leq Knx + \frac{m}{n^3}$$

Consider $ih \in W$ such that $v_{ih} > \overline{mg} + 2mx$. In any realization, at most 1 player who is winning an object is also in a position to affect the price by changing the associated bid. And, the impact of that bid on price is at most $x$. Hence,

$$\sum_{\{ih \in W | v_{ih} > \overline{mg} + 2mx\}} \hat{C}_{ih} \leq x$$

and

$$\sum_{\{ih \in W | v_{ih} > \overline{mg} + 2mx\}} Y_{ih} = 0.$$ 

Finally, consider $ih \notin W$ such that $v_{ih} > \overline{mg} + 2mx$. Then, by deviating to $b_{ih} = v_{ih}$, $i$ raises the price on at most $m - 1$ previous units by at most $x$. But, $i$ also wins an extra object at price at most $\overline{mg}$. So,

$$\hat{B}_{ih} - \hat{C}_{ih} \geq v_{ih} - \overline{mg} - (m - 1)x \geq \frac{v_{ih} - p}{2} = \frac{Y_{ih}}{2}.$$
Since we are in equilibrium
\[ 0 \geq E \left( \sum_{i,h} \hat{B}_{ih} - \hat{C}_{ih} \right) \]
\[ = E \left( E \left( \sum_{\{ih \in W | v_{ih} > u_{y} + 2m \xbar{r} \}} \hat{B}_{ih} - \hat{C}_{ih} \bigg| x \right) \right) + E \left( E \left( \sum_{\{ih \in W | v_{ih} > u_{y} + 2m \xbar{r} \}} \hat{B}_{ih} - \hat{C}_{ih} \bigg| x \right) \right) \]
\[ + E \left( E \left( \sum_{\{ih | v_{ih} \in [c_{g}, u_{y} + 2m \xbar{r}] \}} \hat{B}_{ih} - \hat{C}_{ih} \bigg| x \right) \right) \]
\[ \geq E \left( E \left( \sum_{\{ih | v_{ih} > u_{y} + 2m \xbar{r} \}} \frac{Y_{ih}}{2} \bigg| x \right) \right) - E(x) - E \left( E \left( \sum_{\{ih | v_{ih} \in [c_{g}, u_{y} + 2m \xbar{r}] \}} (m - 1)x \bigg| x \right) \right) \]
\[ \geq E \left( E \left( \sum_{\{ih | v_{ih} > u_{y} + 2m \xbar{r} \}} \frac{Y_{ih}}{2} \bigg| x \right) \right) - E(x) - E \left( (Kn + \frac{m}{n}) (m - 1)x \right) \]
\[ \geq E \left( E \left( \sum_{\{ih | v_{ih} > u_{y} + 2m \xbar{r} \}} \frac{Y_{ih}}{2} \bigg| x \right) \right) - 2E(x) - K'' n E(x^2) \text{ (for } n \text{ SL)} \]

(A.28)

Let \( H \) be the cumulative for \( x \). By Lemmas 6 and 8, \( H(x) \leq \frac{2}{n^x - 0.2x^2} \) for all \( x \). Hence,

\[
 nE(x^2) = \int_0^1 nx^2 dH(x) \\
 = \int_0^1 n2x[1 - H(x)]dx \\
 \leq \int_0^{1/n} 2nx dx + \int_{1/n}^1 2nx \frac{2}{n^{2-\alpha_2}x^2} dx \\
 = nx^2\left[1 - \frac{4}{n^{1-\alpha_2}} \right] + \frac{4}{n^{1-\alpha_2}} \int_{1/n}^1 \frac{1}{x} dx \\
 \leq 1 + 4 \log n \\
 \leq \frac{1}{n^{1-\alpha_2}} \text{ (for } n \text{ SL)}.
\]
And, \( E(x) \leq \frac{1}{n^{1-\alpha_1}} \) as well (a simple integration by parts). So, (A.28) yields

\[
E \left( E \left( \sum_{ih|v_{ih} > ug + 2mx} Y_{ih} \bigg| x \right) \right) \leq 2 \left( \frac{1}{n^{1-\alpha_1}} + K'' \frac{1}{n^{1-\alpha_1}} \right)
\]
\[
= K'' \frac{1}{n^{1-\alpha_1}}
\]

But then,

\[
E \left( \sum_{ih} Y_{ih} \right) = E \left( E \left( \sum_{ih|v_{ih} > ug + 2mx} Y_{ih} \bigg| x \right) \right) + E \left( E \left( \sum_{v_{ih} \in [cg, ug + 2mx]} Y_{ih} \bigg| x \right) \right)
\]
\[
\leq K'' \frac{1}{n^{1-\alpha_1}} + E \left( Kn_x + \frac{m}{n^3} \right) (2m + 1)x
\]
\[
\leq K'' \frac{1}{n^{1-\alpha_1}} + \frac{2m + 1}{n^3} m E(x) + K''' E(n.x^2)
\]
\[
\leq \frac{1}{2n^{1-\alpha}} \text{ for } n SL
\]

Arguing analogously for sellers, \( E(\sum_{ih} Y_{ih}) \leq \frac{1}{2n^{1-\alpha}} \). Hence, the expected sum of consumer and producer surplus is within \( 1/n^{1-\alpha} \) of that achieved by the Walrasian outcome, and we are done. \( Q.E.D. \)
REFERENCES


